Mathematics 204

Spring 2010

Exam II

[1] Your Printed Name: Dr. Grow

[1] Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. Exam II consists of this cover page, 7 pages of problems containing 7 numbered problems, and a short table of Laplace transform formulas.
4. Once the exam begins, you will have 60 minutes to complete your solutions.
5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals, partial fraction decompositions, and determinant computations must be done by hand.
6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
7. The symbol [14] at the beginning of a problem indicates the point value of that problem is 14. The maximum possible score on this exam is 100.

	0	1	2	3	4	5	6	7	**Sum**
points earned									
maximum points	2	12	16	16	12	14	12	16	100

1.[12] Solve the initial value problem: $y'' - 4y' + 3y = 0, \quad y(0) = -2, \; y'(0) = 5.$

$y = e^{mx}$ leads to $m^2 - 4m + 3 = 0 \Rightarrow (m-3)(m-1) = 0 \Rightarrow m = 1$ or 3.

$y_1 = e^{x}, \; y_2 = e^{3x}$. $W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = 3e^{4x} - e^{4x} = 2e^{4x} \neq 0$.

Therefore $y_1 = e^{x}, \; y_2 = e^{3x}$ form a fundamental set of solutions.

The general solution is $y = c_1 e^{x} + c_2 e^{3x}$. Then $y' = c_1 e^{x} + 3c_2 e^{3x}$.

want

$$-2 = y(0) = c_1 + c_2$$
$$5 = y'(0) = c_1 + 3c_2$$

Subtracting equations yields $5 - (-2) = 3c_2 - c_2 \Rightarrow 7 = 2c_2 \Rightarrow c_2 = \frac{7}{2}$.

$\therefore \; c_1 = -2 - c_2 = -2 - \frac{7}{2} = -\frac{11}{2}$.

$\therefore \; \boxed{y(x) = -\frac{11}{2} e^{x} + \frac{7}{2} e^{3x}}$ solves the I.V.P.

2.[16] Find the general solution of the differential equation $y''' - y = 5e^x$. You may find useful the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

$y = e^{mx}$ in $y''' - y = 0$ leads to $m^3 - 1 = 0 \Rightarrow (m-1)(m^2+m+1) = 0$

$\Rightarrow m = 1$ or $m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The general solution to $y''' - y = 0$ is

$$y_c = c_1 e^x + e^{-\frac{x}{2}}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_4 \sin\left(\frac{\sqrt{3}}{2}x\right)\right).$$

$D-1$ annihilates e^x. Then the problem to be solved is $(D^3-1)y = 5e^x$ so

$(D-1)(D^3-1)y = (D-1)(5e^x) = 0$. Substituting $y = e^{mx}$ in this equation leads to

$(m-1)(m^3-1) = 0 \Rightarrow (m-1)^2(m^2+m+1) = 0 \Rightarrow m = 1$ (multiplicity two), $m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

The general solution of $(D-1)(D^3-1)y = 0$ is

$$y = \underline{c_1 e^x} + c_2 x e^x + \underbrace{e^{-\frac{x}{2}}\left(c_3 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_4 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)}.$$

Since the underlined terms appear in y_c, a candidate for $y_p = cxe^x$ where c is a constant to be determined. Then

$$y_p' = cxe^x + ce^x = c(x+1)e^x$$

$$y_p'' = c(x+1)e^x + ce^x = c(x+2)e^x$$

$$y_p''' = c(x+2)e^x + ce^x = c(x+3)e^x.$$

We want $y_p''' - y_p = 5e^x$ so $c(x+3)e^x - cxe^x = 5e^x \Rightarrow 3ce^x = 5e^x$.

Thus $c = \frac{5}{3}$ and so $y_p = \frac{5}{3}xe^x$. The general solution of $y''' - y = 5e^x$ is

$$y = y_c + y_p \Rightarrow \boxed{y(x) = c_1 e^x + e^{-\frac{x}{2}}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right) + \frac{5}{3}xe^x}$$

where c_1 and c_2 are arbitrary constants.

3. (a) [5] Find the general solution of $x^2y''-2xy'+2y=0$ on the interval $0<x<\infty$.

$y=x^r$ leads to $r(r-1)-2r+2=0 \Rightarrow r^2-3r+2=0 \Rightarrow (r-1)(r-2)=0$
$\Rightarrow r=1$ or 2. Then $y_1(x)=x$, $y_2(x)=x^2$. $W(y_1,y_2)(x)=\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0$
if $0<x<\infty$. Therefore the general solution is

$$\boxed{y(x)=c_1x+c_2x^2}.$$

where c_1 and c_2 are constants.

(b) [11] Find the general solution of the nonhomogeneous equation $x^2y''-2xy'+2y=x^2$ on the interval $0<x<\infty$, given that $y_1(x)=x$ and $y_2(x)=x^2$ form a fundamental set of solutions to the associated homogeneous equation on the interval $0<x<\infty$.

Normalizing the nonhomogeneous DE yields $y''-\frac{2}{x}y'+\frac{2}{x^2}y=1$. The variation of parameters formula gives $y_p=u_1(x)y_1(x)+u_2(x)y_2(x)$ where y_1, y_2 is a fundamental set of solutions to the associated homogeneous equation and

$$u_1=\int -\frac{f y_2}{W}dx=\int -\frac{1\cdot x^2}{x^2}dx=\int -1dx=-x+\cancel{c}^{\,0}$$

and

$$u_2=\int \frac{f y_1}{W}dx=\int \frac{1\cdot x}{x^2}dx=\int \frac{1}{x}dx=\ln(x)+\cancel{c}^{\,0}.$$

Therefore a particular solution on the nonhomogeneous equation is

$$y_p(x)=-x\cdot x+x^2\cdot\ln(x).$$

The general solution of the nonhomogeneous equation is

$$y=y_c+y_p=c_1x+c_2x^2-x^2+x^2\ln(x)$$

or

$$\boxed{y=c_1x+\tilde{c}_2x^2+x^2\ln(x)} \qquad (\tilde{c}_2=c_2-1)$$

where $c_1, \tilde{c}_2$ are arbitrary constants.

4.[12] A certain spring hangs vertically from a rigid support. When a 3 pound mass is attached to the end of the spring, the mass stretches the spring 2 feet. Suppose the mass is pulled down 2 additional feet from the rest position and then released. Assuming air resistance (or the damping force) at any instant is equal to twice the instantaneous velocity of the mass, write **BUT DO NOT SOLVE**, an initial value problem describing the motion of the mass.

The vertical displacement from equilibrium position of the body is governed by

$$mx'' + \gamma x' + kx = f(t)$$

where m is the mass of the body, γ is the damping coefficient, k is the spring modulus, and $f(t)$ is the external force exerted on the body at time t. Since the body's weight is mg where g is the acceleration of gravity, we have

$$m = \frac{3 \text{ lb.}}{g} = \frac{3 \text{ lb.}}{32 \text{ ft/sec}^2} = \frac{3}{32} \text{ slugs.}$$

At static equilibrium, we have $mg = ks_0$ where $s_0 = 2$ ft. is the distance the spring has been stretched from its unstretched position. Thus

$$k = \frac{3 \text{ lb.}}{s_0} = \frac{3}{2} \text{ lb/ft.}$$

The damping force at any instant is equal to twice the instantaneous velocity $x'(t)$ of the body. Thus the damping coefficient is $\gamma = 2$ lb./(ft./sec.). Since there is no mention of any external force acting on the body, $f(t) = 0$ for all $t > 0$.

Because the body is pulled down 2 feet from the static equilibrium position and then released, $x(0) = 2$ ft. and $x'(0) = 0$ ft./sec. Hence

$$\boxed{\frac{3}{32}x'' + 2x' + \frac{3}{2}x = 0, \quad x(0) = 2, \; x'(0) = 0}$$

is the I.V.P. that describes the motion of the body.

5.[14] Use the definition of the Laplace transform, $\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)\,dt$ for those values of s for which the improper integral converges, to compute the Laplace transform of the function given by

$$f(t)=\begin{cases} 0 & \text{if } 0\le t<1, \\ t-1 & \text{if } 1\le t<\infty. \end{cases}$$

(Caution: No credit will be awarded for any other method.) For which values of s is the Laplace transform of f defined?

$$\begin{aligned}
\mathcal{L}\{f\}(s) &= \int_0^\infty e^{-st}f(t)\,dt \\
&= \int_0^1 e^{-st}f(t)\,dt + \int_1^\infty e^{-st}f(t)\,dt \\
&= \cancel{\int_0^1 e^{-st}\,0\,dt}^{\,0} + \int_1^\infty e^{-st}(t-1)\,dt. \\
&= \lim_{M\to\infty}\int_1^M e^{-st}(t-1)\,dt.
\end{aligned}$$

We use integration by parts with $U=t-1$ and $dV=e^{-st}dt$. Then $dU=dt$ and $V=\frac{e^{-st}}{-s}$ so

$$\int_1^M e^{-st}(t-1)\,dt = (t-1)\frac{e^{-st}}{-s}\Bigg|_{t=1}^{M} - \int_1^M \frac{e^{-st}}{-s}\,dt = (t-1)\frac{e^{-st}}{-s}\Bigg|_{t=1}^{M} - \frac{e^{-st}}{s^2}\Bigg|_{t=1}^{M}$$

$= \frac{(M-1)}{-se^{Ms}} - \frac{1}{s^2e^{Ms}} + \frac{e^{-s}}{s^2}$. If $s>0$ then $\frac{1}{s^2e^{Ms}}\to 0$ as $M\to\infty$.

Also l'Hospital's rule yields $\lim_{M\to\infty}\frac{(M-1)}{-se^{Ms}} = \lim_{M\to\infty}\frac{1}{-s^2e^{Ms}}=0$. Consequently

$$\mathcal{L}\{f\}(s) = \lim_{M\to\infty}\left(\frac{M-1}{-se^{Ms}} - \frac{1}{s^2e^{Ms}} + \frac{e^{-s}}{s^2}\right) = \boxed{\frac{e^{-s}}{s^2}\ \text{provided } s>0}.$$

If $s<0$ then the above computations show that the improper integral defining $\mathcal{L}\{f\}(s)$ is divergent. Clearly, if $s=0$, then $\int_1^\infty e^{-st}(t-1)\,dt = \int_1^\infty (t-1)\,dt$ is divergent. Therefore the Laplace transform of f is defined only if $\boxed{s>0}$.

6.[12] Find the inverse Laplace transform of $F(s)=\dfrac{s}{(s-2)(s-3)(s-6)}$.

$$\frac{s}{(s-2)(s-3)(s-6)} \overset{\text{P.F.D.}}{=} \frac{A}{s-2}+\frac{B}{s-3}+\frac{C}{s-6}$$ is the partial fraction decomposition of $F(s)$. Here A, B, and C are appropriate constants. Then

$$s = (s-2)(s-3)(s-6)\left[\frac{A}{s-2}+\frac{B}{s-3}+\frac{C}{s-6}\right]$$

$$(*) \qquad s = A(s-3)(s-6)+B(s-2)(s-6)+C(s-2)(s-3).$$

To find A, set $s=2$ in $(*)$: $2 = A(-1)(-4) \Rightarrow A = 1/2$.

To find B, set $s=3$ in $(*)$: $3 = B(1)(-3) \Rightarrow B = -1$.

To find C, set $s=6$ in $(*)$: $6 = C(4)(3) \Rightarrow C = 1/2$.

Therefore

$$F(s) = \frac{1/2}{s-2} - \frac{1}{s-3} + \frac{1/2}{s-6}$$

So

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{1/2}{s-6}\right\}$$

$$= \boxed{\tfrac{1}{2}e^{2t} - e^{3t} + \tfrac{1}{2}e^{6t}}.$$

7.[16] Use the Laplace transform to solve the initial value problem: $y'' + 4y = 8t$, $y(0) = 0$, $y'(0) = 6$. (Caution: No credit will be awarded for any other method.)

We take the Laplace transform of both sides of the differential equation:

$$\mathcal{L}\{y'' + 4y\}(s) = \mathcal{L}\{8t\}(s)$$

$$\mathcal{L}\{y''\}(s) + 4\mathcal{L}\{y\}(s) = \frac{8}{s^2}$$

$$s^2\mathcal{L}\{y\}(s) - s\cancelto{0}{y(0)} - \cancelto{6}{y'(0)} + 4\mathcal{L}\{y\}(s) = \frac{8}{s^2}$$

$$(s^2 + \quad)\mathcal{L}\{y\}(s) = 6 + \frac{8}{s^2}$$

$$\mathcal{L}\{y\}(s) = \frac{6}{s^2+4} + \frac{8}{s^2(s^2+4)}$$

The partial fraction decomposition of $\frac{8}{s^2(s^2+4)}$ is $\frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$.

Therefore $8 = s^2(s^2+4)\left[\frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}\right]$ or

$$(*) \qquad 8 = As(s^2+4) + B(s^2+4) + (Cs+D)s^2.$$

To find B, set $s=0$ in $(*)$: $8 = B(4) \Rightarrow B = 2$.

To find C and D, set $s = 2i$ in $(*)$: $8 = (2iC + D)(-4)$

$$8 + 0i = -4D - 8Ci$$

Therefore $8 = -4D$ and $0 = -8C$ so $D = -2$ and $C = 0$.

To find A, set $s=1$ in $(*)$: $8 = A(5) + B(5) + C + D = 5A + 10 + 0 - 2$ so $A = 0$.

$$\therefore\ y(t) = \mathcal{L}^{-1}\left\{\frac{6}{s^2+4} + \frac{8}{s^2(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{s^2+4} + \frac{2}{s^2} - \frac{2}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2} + \frac{4}{s^2+4}\right.$$

$$\Rightarrow \boxed{y(t) = 2t + 2\sin(2t)}.$$

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. 1	$\frac{1}{s}$
2. e^{at}	$\frac{1}{s-a}$
3. t^n	$\frac{n!}{s^{n+1}}, \quad n = 1, 2, 3....$
4. $\sin(kt)$	$\frac{k}{s^2+k^2}$
5. $\cos(kt)$	$\frac{s}{s^2+k^2}$
6. $f'(t)$	$sF(s) - f(0)$
7. $f''(t)$	$s^2F(s) - sf(0) - f'(0)$