Mathematics 204

Spring 2012

Exam II

Your Printed Name: Dr. Grow

Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam II consists of this cover page, 6 pages of problems containing 6 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [17] at the beginning of a problem indicates the point value of that problem is 17. The maximum possible score on this exam is 100.

problem	1	2	3	4	5	6	**Sum**
points earned							
maximum points	17	17	16	17	17	16	100

1.[17] Find the general solution of each differential equation.

(a) $y^{(5)} + 4y''' = 0$

$y = e^{rt}$ leads to $r^5 + 4r^3 = 0 \Rightarrow r^3(r^2+4) = 0 \Rightarrow r = 0$ (multiplicity three)
$r = \pm 2i$

$\boxed{y(t) = c_1 + c_2 t + c_3 t^2 + c_4 \cos(2t) + c_5 \sin(2t)}$ is the general solution where $c_1, \ldots, c_5$ are arbitrary constants.

(b) $t^2 y'' - 2ty' + 3y = 0$

$y = t^m$ leads to $m(m-1) - 2m + 3 = 0 \Rightarrow m^2 - 3m + 3 = 0$.

$m = \dfrac{3 \pm \sqrt{9-12}}{2} = \dfrac{3}{2} \pm i\dfrac{\sqrt{3}}{2}$. A fundamental set of solutions to the DE on

$(0,\infty)$ is $t^{\lambda}\cos(\mu \ln t) = t^{3/2}\cos\left(\frac{\sqrt{3}}{2}\ln t\right)$ and $t^{\lambda}\sin(\mu \ln t) = t^{3/2}\sin\left(\frac{\sqrt{3}}{2}\ln t\right)$.

Therefore

$$\boxed{y = t^{3/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}\ln t\right) + c_2\sin\left(\frac{\sqrt{3}}{2}\ln t\right)\right]}$$

is the general solution on the interval $0 < t < \infty$; here c_1 and c_2 are arbitrary constants.

2.[17] Given that $y(t)=c_1t+c_2e^{-t}$ is the general solution of the homogeneous equation $(t+1)y''+ty'-y=0$ on the interval $t>-1$, find the general solution of the nonhomogeneous equation $(t+1)y''+ty'-y=(t+1)^2$ for $t>-1$.

Normalizing the DE, we have the equivalent equation $y''+\frac{t}{1+t}y'-\frac{1}{1+t}y=1+t$.

A particular solution has the form $y_p(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$ where $y_1(t)=t$, $y_2(t)=e^{-t}$, $W=\begin{vmatrix} t & e^{-t} \\ 1 & -e^{-t}\end{vmatrix}=-(t+1)e^{-t}$,

$$u_1(t)=\int -\frac{y_2g}{W}\,dt=\int\frac{-e^{-t}(1+t)}{-(t+1)e^{-t}}\,dt=\int 1\,dt=t+\cancel{c}^{\,0},$$

and

$$u_2(t)=\int\frac{y_1g}{W}\,dt=\int\frac{t(t+1)\,dt}{-(t+1)e^{-t}}=\int\underbrace{-t}_{v}\,\underbrace{e^{t}dt}_{dv}=-te^{t}-\int e^{t}(-dt)$$

$$=(-t+1)e^{t}+\cancel{c}^{\,0}.$$

Therefore $y_p(t)=t\cdot t+(1-t)e^{t}\cdot e^{-t}=t^2-t+1$. The general solution is

$$y=y_c+y_p=c_1t+c_2e^{-t}+t^2-t+1=\boxed{\tilde{c}_1t+c_2e^{-t}+t^2+1}$$

where $\tilde{c}_1$ and c_2 are arbitrary constants.

An alternate way to get a particular solution is to use the method of undetermined coefficients with a trial solution of the form $y_p=At^2+Bt+C$ (since $f(t)=(t+1)^2=t^2+2t+1$). Then $y_p'=2At+B$ and $y_p''=2A$. We want to choose A, B, and C so that $(t+1)y_p''+ty_p'-y_p=(t+1)^2$. Substituting yields $(t+1)2A+t(2At+B)-(At^2+Bt+C)=t^2+2t+1$. Simplifying produces

$$At^2+2At+(2A-C)=t^2+2t+1.$$

Therefore $A=1$, $C=1$, and B is arbitrary. That is, $y_p=t^2+Bt+1$ where B is arbitrary. Finish as above.

3.[16] [In the following problem, assume that the acceleration of gravity is 32 feet per second per second.] A body weighing 8 pounds hangs from a vertical spring attached to a rigid support inside a near-vacuum tank. At its equilibrium position, the body stretches the spring 3 inches from its natural length. The viscous damping force is 0.1 pound when the speed of the body is 1000 feet per second. Suppose that the body is displaced an additional 2 inches above the equilibrium position and then released.

(a) If no external forces act on the body, set up, **BUT DO NOT SOLVE,** an initial value problem that models the motion of the body.

If $u(t)$ denotes the vertical displacement (in feet) of the body from the static equilibrium position at time t (in seconds), then $mu'' + \gamma u' + ku = f(t)$ models the body's motion.

8 pounds = weight = $mg = 32m$ so $m = 1/4$ slug.

0.1 pounds = damping force when speed is 1000 ft/sec $= \gamma(1000)$ so $\gamma = 0.0001$.

8 pounds = weight = spring force at static equilibrium $= k(\frac{1}{4}\text{ft.})$ so $k = 32$.

If no external forces act on the body then $f(t) = 0$ for all $t \geq 0$. Therefore an IVP modeling the body's motion is

$$\boxed{\frac{1}{4}u'' + \frac{1}{10{,}000}u' + 32u = 0, \quad u(0) = -\frac{1}{6}, \quad u'(0) = 0}$$

Note: The initial displacement of "2 inches upward from equilibrium" means $u(0) = -\frac{1}{6}$ feet.
"Released" means initial velocity is zero.

(b) Suppose that the body is acted upon by an external downward force of $5\cos(\omega t)$ pounds. Does there exist a value of the frequency ω which will cause resonance? If yes, find the resonant frequency ω. Otherwise, explain why there is no such frequency.

Ans.1 Pure resonance occurs only when no damping forces act on the body. Since $\gamma = \frac{1}{10{,}000} > 0$, pure resonance cannot occur in this situation. I.e there is no pure resonant frequency.

Ans.2 For lightly damped systems, practical resonance occurs when the forcing term is periodic with frequency ω close to $\omega_0 = \sqrt{\frac{k}{m}}$. Since $\gamma = \frac{1}{10{,}000}$, the system is lightly damped. Therefore if $\omega = \omega_0 = \sqrt{\frac{32}{1/4}} = \sqrt{128} = \boxed{8\sqrt{2}}$, practical resonance occurs; i.e. the amplitude of the motion is quite large.

4.[17] Find the general solution of the differential equation $y^{(4)} - 6y''' = \overbrace{3+\cos(x)}^{g(x)}$.

$y = e^{rx}$ in $y^{(4)} - 6y''' = 0$ leads to $r^4 - 6r^3 = 0$ so $r^3(r-6) = 0$ and hence $r = 0$ (with multiplicity 3) and $r = 6$ are the roots of the characteristic equation. Thus, $y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{6x}$ is the general solution of $y^{(4)} - 6y''' = 0$.

Since $g_1(x) = 3$, the method of undetermined coefficients suggests a trial particular solution term of the form $y_{p_1}(x) = x^s A$ where A is a constant to be determined and s is the multiplicity of 0 as a root of the characteristic equation; i.e. $s = 3$. Also, $g_2(x) = \cos(x)$ so the method of undetermined coefficients suggests a trial particular solution term of the form $y_{p_2}(x) = x^s(B\cos(x) + C\sin(x))$ where B and C are constants to be determined and $\alpha + i\beta = 0 + 1\cdot i$ has multiplicity s as a root of the characteristic equation; i.e. $s = 0$. Therefore

$$y_p = y_{p_1} + y_{p_2} = Ax^3 + B\cos(x) + C\sin(x)$$

where A, B, and C are constants to be determined so $y_p^{(4)} - 6y_p''' = 3 + \cos(x)$.

But $y_p' = 3Ax^2 - B\sin(x) + C\cos(x)$, $y_p'' = 6Ax - B\cos(x) - C\sin(x)$, $y_p''' = 6A + B\sin(x) - C\cos(x)$, and $y_p^{(4)} = B\cos(x) + C\sin(x)$. Substituting these expressions in the nonhomogeneous DE gives

$$B\cos(x) + C\sin(x) - 6\left(6A + B\sin(x) - C\cos(x)\right) = 3 + \cos(x)$$

or simplifying,

$$-36A + (B+6C)\cos(x) + (C-6B)\sin(x) = 3 + 1\cdot\cos(x) + 0\cdot\sin(x).$$

Equating like coefficients, $-36A = 3$, $B + 6C = 1$, and $C - 6B = 0$. It follows easily that $A = -\frac{1}{12}$, $B = \frac{1}{37}$, and $C = \frac{6}{37}$. Thus, a particular solution is

$$y_p = -\frac{x^3}{12} + \frac{1}{37}\cos(x) + \frac{6}{37}\sin(x).$$

The general solution is $y = y_c + y_p$ or

$$\boxed{y = c_1 + c_2 x + c_3 x^2 + c_4 e^{6x} - \frac{x^3}{12} + \frac{1}{37}\cos(x) + \frac{6}{37}\sin(x)}$$

5.[17] Use the definition of the Laplace transform,

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

for those values of s for which this improper integral converges, to find the Laplace transform of the function

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 2, \\ e^t & \text{if } t \ge 2. \end{cases}$$

For which values of s is the Laplace tranform of f defined?

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^2 f(t)e^{-st}dt + \int_2^\infty f(t)e^{-st}dt = \int_0^2 te^{-st}dt + \int_2^\infty e^t\cdot e^{-st}dt$$

But $\displaystyle\int_0^2 \underbrace{t}_{U}\,\underbrace{e^{-st}dt}_{dV} = \frac{te^{-st}}{-s}\Big|_0^2 - \int_0^2 \frac{e^{-st}}{-s}dt = \frac{te^{-st}}{-s}\Big|_{t=0}^{2} - \frac{e^{-st}}{s^2}\Big|_{t=0}^{2} = -\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2}$,

and $\displaystyle\int_2^\infty e^t\cdot e^{-st}dt = \lim_{M\to\infty}\int_2^M e^{(1-s)t}dt = \lim_{M\to\infty}\left(\frac{e^{(1-s)t}}{1-s}\Big|_{t=2}^{M}\right) = \lim_{M\to\infty}\left(\frac{e^{(1-s)M} - e^{2(1-s)}}{1-s}\right)$

Therefore the improper integral converges if and only if $s>1$. That is, the Laplace transform of f is defined for $\boxed{s>1}$. Since $\displaystyle\lim_{M\to\infty} e^{(1-s)M} = 0$ when $s>1$, it follows that

$$\int_2^\infty e^t\cdot e^{-st}dt = \frac{e^{-2s+2}}{s-1},$$

and consequently

$$\mathcal{L}\{f\}(s) = \frac{-2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s+2}}{s-1}$$

$$= \boxed{\frac{1}{s^2}\left(1-(2s+1)e^{-2s}\right) + \frac{e^{-2s+2}}{s-1}} \quad \text{if } s>1.$$

6.[16] Use the method of Laplace transforms to solve the initial value problem

$$y^{(4)} - 4y = 0, \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = -2,\ y'''(0) = 0.$$

[Caution: No credit will be awarded for any other method of solution.]

Let $y = y(t)$ be a solution of the IVP. Then $y^{(4)}(t) - 4y(t) = 0$ for all $t \geq 0$, as well as satisfying the initial conditions. Taking the Laplace transform of the functions in the identity yields

$$\mathcal{L}\{y^{(4)}\}(s) - 4\mathcal{L}\{y\}(s) = \mathcal{L}\{0\}(s).$$

Using formula 5 in the short table of Laplace transforms (with $n=4$) yields

7 pts. to here

$$s^4\mathcal{L}\{y\}(s) - s^3\overset{1}{\cancel{y(0)}} - s^2\overset{0}{\cancel{y'(0)}} - s\overset{-2}{\cancel{y''(0)}} - \overset{0}{\cancel{y'''(0)}} - 4\mathcal{L}\{y\}(s) = 0.$$

Applying the initial conditions and rearranging gives

$$(s^4 - 4)\mathcal{L}\{y\}(s) = s^3 - 2s$$

$$\mathcal{L}\{y\}(s) = \frac{s^3 - 2s}{s^4 - 4}$$

12 pts. to here

$$\mathcal{L}\{y\}(s) = \frac{s(s^2-2)}{(s^2-2)(s^2+2)} = \frac{s}{s^2+2}.$$

Therefore

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} = \boxed{\cos(\sqrt{2}\,t)}$$

pts. to here

by formula 4 in the short table of Laplace transforms.

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\frac{1}{s-a}$
2. t^n	$\frac{n!}{s^{n+1}}$, $n = 0,1,2,3....$
3. $\sin(bt)$	$\frac{b}{s^2+b^2}$
4. $\cos(bt)$	$\frac{s}{s^2+b^2}$
5. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - ... - f^{(n-1)}(0)$
6. $e^{ct} f(t)$	$F(s-c)$
7. $u_c(t) f(t-c)$	$e^{-cs} F(s)$