Mathematics 204

Fall 2011

Exam III

Your Printed Name: Dr. Grow

Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam III consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, work must be shown on integration, partial fraction, and matrix computations.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [12] at the beginning of a problem indicates the point value of that problem is 12. The maximum possible score on this exam is 100.

	1	2	3	4	5	**Sum**
points earned						
maximum points	12	22	22	22	22	100

1.[12] If $\mathbf{A}(t)=\begin{pmatrix} \sin(t) & \cos(t) \\ -\cos(t) & \sin(t) \end{pmatrix}$, compute the derivative of the inverse matrix, $\frac{d\left(\mathbf{A}^{-1}(t)\right)}{dt}$.

$$A^{-1}(t) = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{\sin^2(t)+\cos^2(t)}\begin{bmatrix} \sin(t) & -\cos(t) \\ \cos(t) & \sin(t) \end{bmatrix}$$

$$A^{-1}(t) = \begin{bmatrix} \sin(t) & -\cos(t) \\ \cos(t) & \sin(t) \end{bmatrix}$$

$$\frac{d}{dt}A^{-1}(t) = \begin{bmatrix} \frac{d}{dt}\sin(t) & \frac{d}{dt}(-\cos(t)) \\ \frac{d}{dt}(\cos(t)) & \frac{d}{dt}(\sin(t)) \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

2.[22] Solve the initial value problem $x' = x + y,\ y' = 4x + y,\ x(0) = 2,\ y(0) = 0.$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} .$$

$\vec{x} = \vec{k}e^{\lambda t}$ in $\vec{x}' = A\vec{x}$ leads to $\lambda\vec{k} = A\vec{k}$,

i.e. the eigenvalue equation for A. The eigenvalues of A satisfy

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda-1) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$$

Eigenvalues	Eigenvectors
$\lambda_1 = 3$	$\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$\lambda_2 = -1$	$\vec{k}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

An eigenvector $\vec{k}^{(1)}$ of A corresponding to $\lambda = 3$ satisfies $(A - \lambda I)\vec{k} = \vec{0}$; i.e.

$$\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or equivalently $\begin{cases} -2k_1 + k_2 = 0 \quad \Rightarrow k_2 = 2k \\ \cancel{4k_1 - 2k_2 = 0} \quad \text{Redundant, } -2 \text{ times eqn.} \end{cases}$

Thus $\vec{k}^{(1)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ 2k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} k_1 .$

An eigenvector $\vec{k}^{(2)}$ of A corresponding to $\lambda = -1$ satisfies $(A - \lambda I)\vec{k} = \vec{0}$; i.e.

$$\begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or}$$

equivalently $\begin{cases} 2k_1 + k_2 = 0 \quad \Rightarrow k_2 = -2k_1 \\ \cancel{4k_1 + 2k_2 = 0} \quad \text{Redundant; 2 times eqn. 1.} \end{cases}$

Therefore $\vec{k}^{(2)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -2k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} k_1 .$

Consequently $\vec{x}(t) = c_1 \vec{k}^{(1)} e^{\lambda_1 t} + c_2 \vec{k}^{(2)} e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ is the general solution to $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$ where c_1 and c_2 are arbitrary constants.

We need to choose c_1 and c_2 so that $\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$ It follows that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \vec{x}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} \quad \text{so} \quad \boxed{\begin{aligned} x(t) &= e^{3t} + e^{-t} \\ y(t) &= 2e^{3t} - 2e^{-t} \end{aligned}} \quad \text{solves the IVP.}$$

3.[22] Consider the matrix $\mathbf{A} = \begin{pmatrix} 0 & -4 \\ 1 & \alpha \end{pmatrix}$.

(a) Determine the values of α which will result in the matrix $\mathbf{A}$ having two complex, nonreal eigenvalues.

(b) Solve the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ given that $\alpha = 0$.

(a) The eigenvalues λ of A satisfy $0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & -4 \\ 1 & \alpha-\lambda \end{vmatrix} = (\lambda-\alpha)\lambda + 4 = \lambda^2 - \alpha\lambda + 4$

By the quadratic equation, $\lambda = \dfrac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$ so A will have two complex, nonreal eigenvalues if and only if $\alpha^2 - 16 < 0$, i.e. $\boxed{-4 < \alpha < 4}$.

(b) $\vec{x} = \vec{k}e^{\lambda t}$ in $\vec{x}' = A\vec{x}$ leads to the eigenvalue equation for A: $\lambda\vec{k} = A\vec{k}$.

The eigenvalues λ of A satisfy $0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & -4 \\ 1 & \alpha-\lambda \end{vmatrix} = \lambda^2 - \alpha\lambda + 4$ so when $\alpha = 0$ this becomes $0 = \lambda^2 + 4$. Hence $\lambda = \pm 2i$. An eigenvector $\vec{k}^{(1)}$ of A corresponding to $\lambda_1 = 2i$ satisfies $(A - \lambda I)\vec{k} = \vec{0}$ so $\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or equivalenty $\begin{cases} \cancel{-2ik_1 - 4k_2 = 0} & \text{Redundant; } -2i \text{ times eqn. 2.} \\ k_1 - 2ik_2 = 0 & \Rightarrow k_1 = 2ik_2. \end{cases}$

Therefore $\vec{k}^{(1)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2ik_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2i \\ 1 \end{bmatrix} k_2$. We may take $k_2 = 1$ so $\vec{k}^{(1)} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$.

Then $\vec{x}^{(1)}(t) = \vec{k}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{2it}$ is a complex solution of $\vec{x}' = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}\vec{x}$.

Linearly independent real solutions are $\tilde{\vec{x}}^{(1)}(t) = \mathrm{Re}(\vec{x}^{(1)}(t))$, $\tilde{\vec{x}}^{(2)}(t) = \mathrm{Im}(\vec{x}^{(1)}(t))$.

But $\vec{x}^{(1)}(t) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)(\cos(2t) + i\sin(2t)) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin(2t)\right) + i\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos(2t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(2t)\right)$

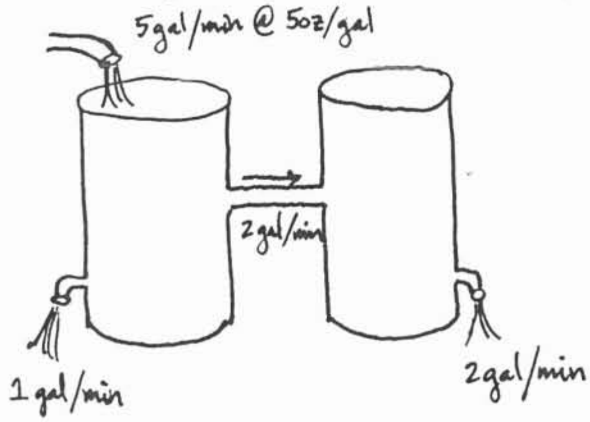
Therefore $\tilde{\vec{x}}^{(1)}(t) = \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix}$, $\tilde{\vec{x}}^{(2)}(t) = \begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix}$ is a F.S.S. for $\vec{x}' = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}\vec{x}$.

The general solution is

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2\begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix}}$$

where c_1 and c_2 are arbitrary constants.

4.[22] Two tanks initially hold 200 gallons of pure water each. A mixture of salt and water at a concentration of 5 ounces per gallon flows into Tank 1 at a rate of 5 gallons per minute. The well-stirred mixture drains from Tank 1 into Tank 2 at the rate of 2 gallons per minute and into the environment at the rate of 1 gallon per minute. Tank 2 is fed only from Tank 1 and the well-stirred mixture in Tank 2 drains into the environment at a rate of 2 gallons per minute. If $Q_1(t)$ and $Q_2(t)$ denote the salt content in ounces in Tanks 1 and 2, respectively, set up, **BUT DO NOT SOLVE**, differential equations and initial conditions that model the flow process.



The volume of solution in Tank 1 at time t minutes is $200 + 5t - 2t - t = 200 + 2t$ gallons.

The volume of solution in Tank 2 at time t minutes is constant: 200 gallons

The net rate of change of salt in Tank 1 is equal to the rate of inflow of salt to Tank 1 minus the rate of outflow of salt from Tank 1. Thus

$$\frac{dQ_1}{dt} = \left(\frac{5\,\text{gal}}{\text{min}}\right)\left(\frac{5\,\text{oz}}{\text{gal}}\right) - \left(\frac{2\,\text{gal}}{\text{min}}\right)\left(\frac{Q_1\,\text{oz}}{(200+2t)\,\text{gal}}\right) - \left(\frac{1\,\text{gal}}{\text{min}}\right)\left(\frac{Q_1\,\text{oz}}{(200+2t)\,\text{gal}}\right).$$

Similarly, for Tank 2 the net rate of change of salt is equal to the rate of inflow of salt into Tank 2 minus the rate of outflow of salt from Tank 2:

$$\frac{dQ_2}{dt} = \left(\frac{2\,\text{gal}}{\text{min}}\right)\left(\frac{Q_1\,\text{oz}}{(200+2t)\,\text{gal}}\right) - \left(\frac{2\,\text{gal}}{\text{min}}\right)\left(\frac{Q_2\,\text{oz}}{200\,\text{gal}}\right).$$

Simplifying and augmenting the initial conditions gives

$$\boxed{\begin{aligned} \frac{dQ_1}{dt} &= 25 - \left(\frac{3}{200+2t}\right)Q_1, \quad Q_1(0) = 0 \\ \frac{dQ_2}{dt} &= \frac{1}{100+t}Q_1 - \frac{1}{100}Q_2, \quad Q_2(0) = 0 \end{aligned}}$$

where Q_1 and Q_2 are in ounces while t is in minutes.

5.[22] Find the solution to the integro-differential equation

$$y'(t)+3\int_0^t e^{-4\tau}y(t-\tau)\,d\tau=1$$

satisfying the initial condition $y(0)=0$.

We rewrite the equation as $y'(t)+3(f*y)(t)=1$ where $f(t)=e^{-4t}$ and $(f*y)(t)=\int_0^t f(\tau)y(t-\tau)\,d\tau$ is the convolution integral of f and y at t.

We take the Laplace transform of both sides of this equation and apply formulas 5, 6 (with $n=1$), and 2 (with $n=0$) in the Laplace transform table to obtain

$$\mathcal{L}\{y'(t)+3(f*y)(t)\}(s)=\mathcal{L}\{1\}(s)$$

$$sY(s)-\cancel{y(0)}^{\,0}+3F(s)Y(s)=\frac{1}{s}$$

$$sY(s)+3\frac{1}{s+4}Y(s)=\frac{1}{s}. \qquad \text{(by formula 1)}$$

Solving for $Y(s)$ yields

$$\left(s+\frac{3}{s+4}\right)Y(s)=\frac{1}{s}$$

$$\frac{s(s+4)+3}{s+4}Y(s)=\frac{1}{s}$$

$$Y(s)=\frac{s+4}{s(s^2+4s+3)}$$

$$Y(s)=\frac{s+4}{s(s+3)(s+1)}.$$

We use a partial fraction decomposition of the rational function of s in the right member of the last equation above:

$$\frac{s+4}{s(s+3)(s+1)}=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{s+1}$$

$$\Rightarrow\quad s+4=A(s+3)(s+1)+Bs(s+1)+Cs(s+3).$$

(OVER)

We set $s=0$ to find A: $\quad 0+4 = A(0+3)(0+1) + B(0)(0+1) + C(0)(0+3)$

$$\frac{4}{3} = A.$$

We set $s=-3$ to find B: $\quad -3+4 = A(0)(-3+1) + B(-3)(-3+1) + C(-3)(0)$

$$\frac{1}{6} = B.$$

We set $s=-1$ to find C: $\quad -1+4 = A(-1+3)(0) + B(-1)(0) + C(-1)(-1+3)$

$$-\frac{3}{2} = C.$$

Therefore $\quad Y(s) = \dfrac{4/3}{s} + \dfrac{1/6}{s+3} - \dfrac{3/2}{s+1}.$

Taking the inverse Laplace transform of both sides yields the solution:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4/3}{s} + \frac{1/6}{s+3} - \frac{3/2}{s+1}\right\}$$

$$\boxed{y(t) = \frac{4}{3} + \frac{1}{6}e^{-3t} - \frac{3}{2}e^{-t}}.$$

(We used formula 1 in the Laplace transform table with $a=0$, $a=-3$, and $a=-1$, respectively.)

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\frac{1}{s-a}$
2. t^n	$\frac{n!}{s^{n+1}}, \quad n = 0,1,2,3....$
3. $\sin(bt)$	$\frac{b}{s^2+b^2}$
4. $\cos(bt)$	$\frac{s}{s^2+b^2}$
5. $(f*g)(t)$	$F(s)G(s)$
6. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - ... - f^{(n-1)}(0)$
7. $e^{ct} f(t)$	$F(s-c)$
8. $u_c(t) f(t-c)$	$e^{-cs} F(s)$

2011 Fall Semester, Math 204 Hour Exam III, Master List

100	卌		59			19	
99	卌 卌					58				18	
98							57						17	
97	卌 卌			56							16	
96	卌 卌 卌 卌 ||||		55	||		15	
95	卌 卌 卌				54							14		
94	卌 卌 ||		53	||||		13	
93	卌 卌 ||||	139 A's	52	|||	41 F's	12	
92	卌 卌 ||||		51	|||		11	
91	卌 卌				50			10		
90	卌 卌					49				9	
89	卌 卌		48				8	
88	卌 卌 卌		47			7	
87	卌						46				6	
86	卌 ||||		45	||		5	
85	卌 卌 卌 |||		44			4	
84	卌 卌				113 Bs	43			3		
83	卌			42				2	
82	卌 卌 卌			41					1	
81	卌 ||		40			0	
80	卌 卌		39				
79						38					
78	卌 卌		37				
77	卌 |||		36				
76	卌					72 C's	35					
75	卌 卌			34				
74							33					
73	卌		32				
72	卌 ||||		31				
71	卌 |||		30				
70	卌		29				
69	卌 ||||		28				
68							27					
67							26					
66	卌 ||||		25				
65	卌		51 D's	24				
64	卌			23				
63						22				
62	卌		21				
61	卌		20					
60							

Number taking exam: 416
Median: 84
Mean: 80.0
Standard Deviation: 15.9

Number receiving A's:	139	33.4%
Number receiving B's:	113	27.2
Number receiving C's:	72	17.3
Number receiving D's:	51	12.3
Number receiving F's:	41	9.9