Mathematics 204

Fall 2012

Exam III

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**

2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.

3. You are **not allowed to use a calculator** on this exam.

4. Exam III consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.

5. Once the exam begins, you will have 60 minutes to complete your solutions.

6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.

7. Express all solutions in real-valued, simplified form.

8. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.

9. The symbol [22] at the beginning of a problem indicates the point value of that problem is 22. The maximum possible score on this exam is 100.

	1	2	3	4	5	**Sum**
points earned						
maximum points	22	20	14	22	22	100

1.(a) [18] Find the solution $y=y(t)$ of the initial value problem $y''+3y'+2y=\delta(t-5)$, $y(0)=0$, $y'(0)=1$.

(b) [4] Which is greater, $y(6)$ or $y(1)$? Justify your answer.

(a) We use the Laplace transform method because the forcing term is a Dirac delta.

$$\mathcal{L}\{y''+3y'+2y\}(s) = \mathcal{L}\{\delta(t-5)\}$$

$$s^2\mathcal{L}\{y\}(s) - s\cancelto{0}{y(0)} - \cancelto{1}{y'(0)} + 3\left(s\mathcal{L}\{y\}(s) - \cancelto{0}{y(0)}\right) + 2\mathcal{L}\{y\}(s) = e^{-5s}$$ (See 6 and 9 in Laplace table)

$$(s^2+3s+2)\mathcal{L}\{y\}(s) = 1 + e^{-5s}$$

$$\mathcal{L}\{y\}(s) = \frac{1}{(s+2)(s+1)} + e^{-5s}\cdot\frac{1}{(s+2)(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+1)} + e^{-5s}\cdot\frac{1}{(s+2)(s+1)}\right\}.$$

$$\frac{1}{(s+2)(s+1)} \overset{\text{P.F.D.}}{=} \frac{A}{s+2} + \frac{B}{s+1} \Rightarrow 1 = A(s+1) + B(s+2).$$

To find A, set $s=-2$: $1 = A(-1) \Rightarrow A=-1$.

To find B, set $s=-1$: $1 = B(1) \Rightarrow B = 1$.

$$\therefore\quad y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{e^{-5s}\cdot\left(\frac{1}{s+1} - \frac{1}{s+2}\right)\right\}$$

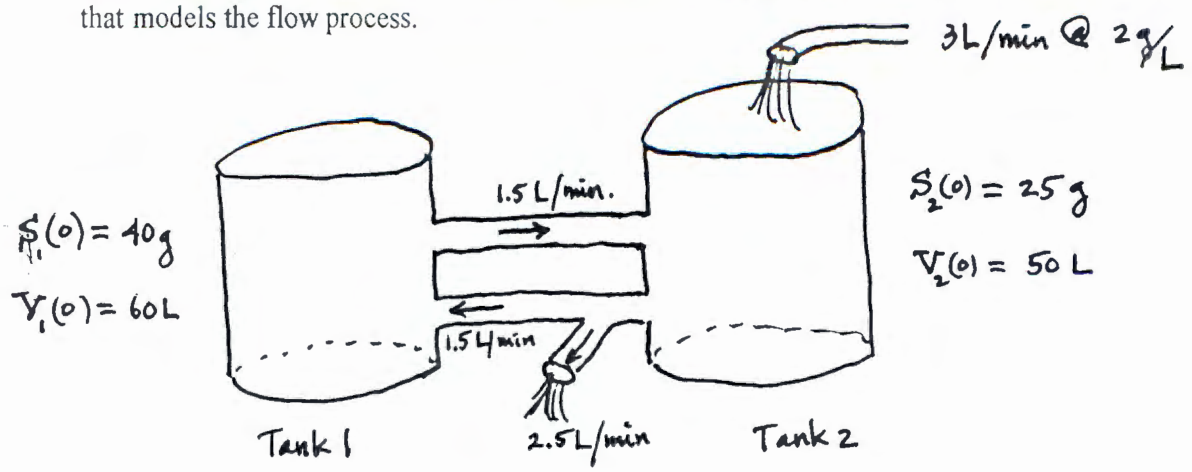
$$\boxed{y(t) = e^{-t} - e^{-2t} + u_5(t)\left(e^{-(t-5)} - e^{-2(t-5)}\right)}$$ (See 1 and 8 in Laplace table)

(b) $$y(1) = e^{-1} - e^{-2} + \overset{0}{\widetilde{u_5(1)}}\left(e^{+4} - e^{+8}\right) = e^{-1} - e^{-2}$$ Same

$$y(6) = e^{-6} - e^{-12} + \underbrace{u_5(6)}_{1}\left(e^{-1} - e^{-2}\right) = e^{-1} - e^{-2} + \left(e^{-6} - e^{-12}\right)$$ positive since $e^{-t}\searrow$.

Therefore $\boxed{y(6) > y(1).}$

2.[20] Consider two interconnected tanks. Tank 1 initially contains 40 grams of sugar dissolved in 60 liters of water and Tank 2 initially contains 50 liters of water with 25 grams of sugar. Water containing 2 grams of sugar per liter flows into Tank 2 at a rate of 3 liters per minute. The well-stirred mixture drains from Tank 2 at a rate of 4 liters per minute, of which some flows into Tank 1 at a rate of 1.5 liters per minute while the remainder leaves the system. The well-stirred mixture in Tank 1 flows back into Tank 2 at a rate of 1.5 liters per minute. If $S_1(t)$ and $S_2(t)$ denote the amounts of sugar at time t in Tanks 1 and 2, respectively, set up, **BUT DO NOT SOLVE,** an initial value problem that models the flow process.



Net rate of change of sugar w.r.t. time = Rate at which sugar enters − Rate at which sugar leaves

Tank 1: $\dfrac{dS_1}{dt} = (1.5\,\cancel{L}/\text{min})\left(\dfrac{S_2(t)\,g}{V_2(t)\,\cancel{L}}\right) - (1.5\,\cancel{L}/\text{min})\left(\dfrac{S_1(t)\,g}{V_1(t)\,\cancel{L}}\right)$

Tank 2: $\dfrac{dS_2}{dt} = (3\,\cancel{L}/\text{min})(2\,g/\cancel{L}) + (1.5\,\cancel{L}/\text{min})\left(\dfrac{S_1(t)\,g}{V_1(t)\,\cancel{L}}\right) - (4.0\,\cancel{L}/\text{min})\left(\dfrac{S_2(t)\,g}{V_2(t)\,L}\right)$

$V_1(t) = V_1(0) = 60$ for all $t \geq 0$.

$V_2(t) = V_2(0) + \frac{1}{2}t = 50 + \frac{t}{2}$ for $t \geq 0$.

$$\boxed{\begin{aligned} \frac{dS_1}{dt} &= -\frac{1.5}{60}S_1 + \frac{1.5}{50+t/2}S_2, \quad && S_1(0) = 40 \\ \frac{dS_2}{dt} &= \frac{1.5}{60}S_1 - \frac{4.0}{50+t/2}S_2 + 6, \quad && S_2(0) = 25 \end{aligned}}$$

(S_1, S_2 in grams, t in minutes)

3.[14] If $A(t)=\begin{pmatrix} 2e^{2t} & e^{-t} \\ 2e^{t} & 3e^{-2t} \end{pmatrix}$, find $\frac{d}{dt}\left(A^{-1}(t)\right)$.

$$A^{-1}(t) = \frac{1}{\det A(t)}\begin{bmatrix} A_{22}(t) & -A_{12}(t) \\ -A_{21}(t) & A_{11}(t) \end{bmatrix} = \frac{1}{6-2}\begin{bmatrix} 3e^{-2t} & -e^{-t} \\ -2e^{t} & 2e^{2t} \end{bmatrix}$$

$$\frac{d}{dt}\left(A^{-1}(t)\right) = \frac{1}{4}\begin{bmatrix} -6e^{-2t} & e^{-t} \\ -2e^{t} & 4e^{2t} \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{3}{2}e^{-2t} & \frac{1}{4}e^{-t} \\ -\frac{1}{2}e^{t} & e^{2t} \end{bmatrix}}$$

4.[22] Solve the integral equation $y(t) = 3t^2 - e^{-t} - \int_0^t y(u) e^{t-u} du$.

Using the definition of the convolution product, $f * g(t) = \int_0^t f(u) g(t-u) du$, we see that $\int_0^t y(u) e^{t-u} du$ is the convolution product of $y(t)$ and e^t.

Therefore the integral equation can be written as

$$y(t) = 3t^2 - e^{-t} - (y * \exp)(t).$$

Taking the Laplace transform of both sides and using formulas 2, 1, and 5 in the Laplace transform table yields:

$$\mathcal{L}\{y\}(s) = \frac{6}{s^3} - \frac{1}{s+1} - \mathcal{L}\{y\}(s) \cdot \frac{1}{s-1}.$$

Rearranging and simplifying

$$\left(1 + \frac{1}{s-1}\right) \mathcal{L}\{y\}(s) = \frac{6}{s^3} - \frac{1}{s+1}$$

$$\frac{s}{s-1} \mathcal{L}\{y\}(s) = \frac{6}{s^3} - \frac{1}{s+1}$$

$$\mathcal{L}\{y\}(s) = \frac{6(s-1)}{s^4} - \frac{s-1}{s(s+1)} = \frac{6}{s^3} - \frac{6}{s^4} - \frac{s-1}{s(s+1)}.$$

The partial fraction decomposition of the last term in the right member is

$\frac{s-1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$ so $s-1 = A(s+1) + Bs$. To find A, set $s=0$: $-1 = A$. To find B set $s=-1$: $-2 = -B$ so $B = 2$.

$$\therefore \quad y(t) = \mathcal{L}^{-1}\left\{ \frac{6}{s^3} - \frac{6}{s^4} - \left(\frac{-1}{s} + \frac{2}{s+1}\right)\right\}$$

$$\boxed{y(t) = 3t^2 - t^3 + 1 - 2e^{-t}}.$$

5.(a) [20] Solve the initial value problem $\mathbf{x}' = \underbrace{\begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix}}_{A}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

(b) [2] Describe the behavior of the solution as $t \to \infty$.

(a) $\vec{x} = \vec{k}e^{rt}$ in $\vec{x}' = A\vec{x}$ leads to $r\vec{k} = A\vec{k}$ so r is an eigenvalue of A and $\vec{k}$ an eigenvector. $0 = \det(A - rI) = \begin{vmatrix} -3-r & -1 \\ 2 & -1-r \end{vmatrix} = (r+1)(r+3)+2$.

$0 = r^2 + 4r + 5$ so $r = \dfrac{-4 \pm \sqrt{16-20}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i$.

An eigenvector $\vec{k}$ of A corresponding $r = -2+i$ satisfies $(A - rI)\vec{k} = \vec{0}$ so

$$\begin{bmatrix} -3-(-2+i) & -1 \\ 2 & -1-(-2+i) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is equivalent to $\begin{cases} (-1-i)k_1 - k_2 = 0 \\ 2k_1 + (1-i)k_2 = 0. \end{cases}$

Note that $-(1-i)$ times the first equation of the system is equal to the second equation. Therefore the seond equation is redundant and the solution is $k_2 = -(1+i)k_1$. Thus $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -(1+i)k_1 \end{bmatrix} = -k_1\begin{bmatrix} -1 \\ 1+i \end{bmatrix}$. Take $k_1 = -1$ so

$$\vec{x}^{(1)}(t) = \vec{k}^{(1)}e^{r_1 t} = \begin{bmatrix} -1 \\ 1+i \end{bmatrix}e^{(-2+i)t} = e^{-2t}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\left(\cos(t) + i\sin(t)\right).$$

A fundamental set of real-valued solutions is

$$\tilde{\vec{x}}^{(1)}(t) = \mathrm{Re}\left(\vec{x}^{(1)}(t)\right) = e^{-2t}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\cos(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(t)\right) = e^{-2t}\begin{bmatrix} -\cos(t) \\ \cos(t) - \sin(t) \end{bmatrix}$$

$$\tilde{\vec{x}}^{(2)}(t) = \mathrm{Im}\left(\vec{x}^{(1)}(t)\right) = e^{-2t}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\sin(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(t)\right) = e^{-2t}\begin{bmatrix} -\sin(t) \\ \cos(t) + \sin(t) \end{bmatrix}.$$

The general solution is $\vec{x}(t) = c_1\tilde{\vec{x}}^{(1)}(t) + c_2\tilde{\vec{x}}^{(2)}(t)$ where c_1 and c_2 are arbitrary constants. Applying the initial condition gives

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1\tilde{\vec{x}}^{(1)}(0) + c_2\tilde{\vec{x}}^{(2)}(0) = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ so $c_1 = 1$ and $c_2 = -1$.

Thus $$\boxed{\vec{x}(t) =} e^{-2t}\begin{bmatrix} -\cos(t) \\ \cos(t) - \sin(t) \end{bmatrix} - e^{-2t}\begin{bmatrix} -\sin(t) \\ \cos(t) + \sin(t) \end{bmatrix} = \boxed{e^{-2t}\begin{bmatrix} \sin(t) - \cos(t) \\ -2\sin(t) \end{bmatrix}}$$

(b) As $t \to \infty$, $\boxed{\vec{x}(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$ since $e^{-2t} \to 0$ as $t \to \infty$ and $\begin{bmatrix} \sin(t) - \cos(t) \\ -2\sin(t) \end{bmatrix}$ is bounded on $0 \le t < \infty$.

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\frac{1}{s-a}$
2. t^n	$\frac{n!}{s^{n+1}}$, $n = 0,1,2,3....$
3. $\sin(bt)$	$\frac{b}{s^2+b^2}$
4. $\cos(bt)$	$\frac{s}{s^2+b^2}$
5. $f*g(t)$	$F(s)G(s)$
6. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - ... - f^{(n-1)}(0)$
7. $e^{ct} f(t)$	$F(s-c)$
8. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
9. $\delta(t-c)$	e^{-cs}

2012 Fall Semester, Math 204 Hour Exam III, Master List

100 卌 III
99 卌
98 卌 卌 I
97 卌 III
96 卌 卌
95 卌 卌 I
94 卌 II
93 卌 I
92 卌 II
91 卌 卌
90 卌 III

91 As

89 卌 III
88 卌 I
87 卌 I
86 卌 IIII
85 II
84 卌 II
83 卌 III
82 卌
81 卌 I
80 卌 II

64 Bs

79 卌 卌
78 卌
77 卌 II
76 卌 I
75 卌 IIII
74 卌
73 卌 IIII
72 IIII
71 卌 I
70 卌

66 Cs

69 卌 II
68 卌
67 卌
66 卌
65 卌 II
64 卌
63 卌 II
62 III
61 III
60 III

50 Ds

59 III
58 IIII
57 卌
56 III
55 IIII
54 IIII
53 III
52 II
51 III
50 卌
49 II
48 I
47 IIII
46 II
45 III
44 III
43 I
42 II
41 IIII
40
39 I
38 I
37
36 I
35
34 I
33 III
32 I
31
30 I
29 III
28
27
26 II
25 II
24 I
23
22 I
21
20 I

81 Fs

19 I
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13 I
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5 I
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2 I
1
0

Number taking exam: 352
Median: 77
Mean: 73.3
Standard Deviation: 20.0

Number receiving A's:	91	25.9%
Number receiving B's:	64	18.2
Number receiving C's:	66	18.8
Number receiving D's:	50	14.2
Number receiving F's:	81	23.0

2012 Fall Semester, Math 204 Hour Exam III
Instructor Grow, Section M

100	II	
99		
98	I	
97		
96		8 As
95		
94	I	
93		
92	I	
91	III	
90		
89	I	
88	I	
87	I	
86	I	
85		9 Bs
84		
83	II	
82		
81	I	
80	II	
79	I	
78		
77	I	
76		
75	I	4 Cs
74		
73	I	
72		
71		
70		
69		
68		
67		
66		
65	I	2 Ds
64		
63	I	
62		
61		
60		

59		
58	II	
57		
56		
55	I	6 Fs
54	I	
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Number taking exam: 29
Median: 81
Mean: 77.2
Standard Deviation: 18.0

Number receiving A's: 8 — 27.6%
Number receiving B's: 9 — 31.0
Number receiving C's: 4 — 13.8
Number receiving D's: 2 — 6.9
Number receiving F's: 6 — 20.7