Mathematics 204

Spring 2012

Exam III

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam III consists of this cover page, 6 pages of problems containing 6 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, work must be shown on integration, partial fraction, and matrix computations.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbols [18] at the beginning of a problem indicate the point value of that problem is 18. The maximum possible score on this exam is 100.

	1	2	3	4	5	6	Sum
points earned							
maximum points	18	18	18	18	10	18	100

1.[18] Find the solution of the initial value problem $y''+4y=2-4u_{\pi/2}(t)$, $y(0)=0$, $y'(0)=0$.

We use the Laplace transform method.

$$\mathcal{L}\{y''+4y\}(s) = \mathcal{L}\{2-4u_{\pi/2}(t)\}(s)$$

Using linearity of the Laplace transform, together with formulas 5 (with $n=2$), 2 (with $n=0$), and 7 (with $f(t-c)=1$) in the table of Laplace transforms, gives

$$s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) + 4\mathcal{L}\{y\}(s) = \frac{2}{s} - \frac{4e^{-\frac{\pi}{2}s}}{s}.$$

Incorporating the initial conditions yields

$$(s^2+4)\mathcal{L}\{y\}(s) = \frac{2}{s} - \frac{4}{s}e^{-\frac{\pi}{2}s}$$

$$\mathcal{L}\{y\}(s) = \frac{2}{s(s^2+4)} - \frac{4}{s(s^2+4)}\cdot e^{-\frac{\pi}{2}s}$$

Then

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s(s^2+4)}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{s(s^2+4)}\cdot e^{-\frac{\pi}{2}s}\right\}.$$

A partial fraction decomposition calculation gives

$$\frac{2}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \quad \text{so} \quad 2 = A(s^2+4) + (Bs+C)s.$$

Taking $s=0$ gives A: $2=4A \Rightarrow A=1/2$. Taking $s=2i$ gives B and C: $2+0i = (B(2i)+C)2i = -4B+2Ci$ so $B=-1/2$ and $C=0$. Therefore

$$\mathcal{L}^{-1}\left\{\frac{2}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s} - \frac{\frac{1}{2}s}{s^2+4}\right\} = \frac{1}{2} - \frac{1}{2}\cos(2t),$$

and using formula 7 in the table of Laplace transforms produces

$$\mathcal{L}^{-1}\left\{\frac{4}{s(s^2+4)}e^{-\frac{\pi}{2}s}\right\} = f\left(t-\tfrac{\pi}{2}\right)u_{\frac{\pi}{2}}(t) \quad \text{where } f(t) = \mathcal{L}^{-1}\left\{\frac{4}{s(s^2+4)}\right\} = 1-\cos(2t).$$

Consequently

$$\boxed{y(t) = \frac{1}{2} - \frac{1}{2}\cos(2t) - \left[1-\cos\left(2\left(t-\tfrac{\pi}{2}\right)\right)\right]u_{\frac{\pi}{2}}(t).}$$

2.[18] Find the solution of the initial value problem $y''+y=\frac{t}{\pi}\delta(t-\pi)$, $y(0)=0$, $y'(0)=-1$.

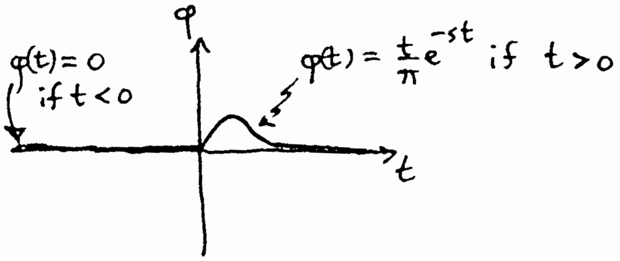
Write your solution as a piecewise defined function and sketch its graph on the interval $0\le t\le 3\pi$.

We use the Laplace transform method.

$$\mathcal{L}\{y''+y\}(s) = \mathcal{L}\{\tfrac{t}{\pi}\delta(t-\pi)\}(s)$$

so
$$s^2\mathcal{L}\{y\}(s) - s\overset{0}{\cancel{y(0)}} - \overset{-1}{\cancel{y'(0)}} + \mathcal{L}\{y\}(s) = \mathcal{L}\{\tfrac{t}{\pi}\delta(t-\pi)\}(s).$$

In order to evaluate the Laplace transform on the right side of the above equation we use the definition of the Laplace transform: $\mathcal{L}\{f\}(s)=\int_0^\infty f(t)e^{-st}dt$ and the sifting property of the Dirac delta (cf. (16) on p. 342 of Boyce and DiPrima):
$\int_{-\infty}^{\infty}\delta(t-t_0)f(t)dt = f(t_0)$ for every bounded, continuous function f on $(-\infty,\infty)$ and every real number t_0. Then



$$\mathcal{L}\{\tfrac{t}{\pi}\delta(t-\pi)\}(s) = \int_0^\infty \delta(t-\pi)\tfrac{t}{\pi}e^{-st}dt$$
$$= \int_{-\infty}^{\infty}\delta(t-\pi)\varphi(t)dt$$
$$= \varphi(\pi) = \frac{\pi}{\pi}e^{-s\pi} = e^{-s\pi}.$$

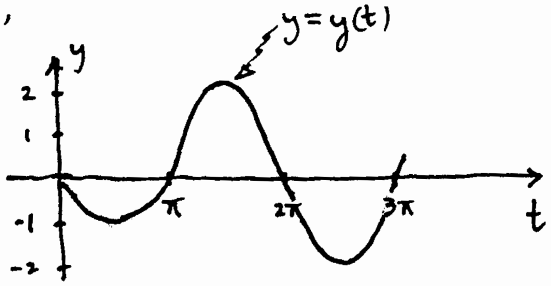
Thus,
$$(s^2+1)\mathcal{L}\{y\}(s) = -1 + e^{-s\pi}$$

so
$$y(t) = \mathcal{L}^{-1}\left\{\frac{-1}{s^2+1} + \frac{1}{s^2+1}\cdot e^{-s\pi}\right\} = -\sin(t) + \sin(t-\pi)u_\pi(t)$$

by formulas 3 (with $b=1$) and 7 (with $c=\pi$ and $f(t-c)=\sin(t-c)$) in the table of Laplace transforms. Since $u_\pi(t)=\begin{cases}0 & \text{if } t<\pi,\\ 1 & \text{if } \pi\le t,\end{cases}$

$$y(t)=\begin{cases}-\sin(t) & \text{if } t<\pi,\\ -\sin(t)+\sin(t-\pi) & \text{if } \pi\le t,\end{cases}$$

$$\boxed{y(t)=\begin{cases}-\sin(t) & \text{if } t<\pi,\\ -2\sin(t) & \text{if } \pi\le t.\end{cases}}$$



3.[18] Solve the integral equation $y(t)+5\int_0^t y(t-\tau)\cos(2\tau)\,d\tau = e^{-3t}$.

We use the Laplace transform method.

$$\mathcal{L}\left\{ y(t) + 5\int_0^t y(t-\tau)\cos(2\tau)\,d\tau \right\}(s) = \mathcal{L}\{e^{-3t}\}(s)$$

$$(*) \qquad \mathcal{L}\{y\}(s) + 5\mathcal{L}\left\{\int_0^t y(t-\tau)\cos(2\tau)\,d\tau\right\}(s) = \frac{1}{s+3}.$$

But $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$ so $\int_0^t y(t-\tau)\cos(2\tau)\,d\tau$ is the convolution product of $f(t)=y(t)$ and $g(t)=\cos(2t)$. Using formulas 8 and 4 (with $b=2$) gives

$$\mathcal{L}\left\{\int_0^t y(t-\tau)\cos(2\tau)\,d\tau\right\}(s) = F(s)G(s) = \mathcal{L}\{y\}(s)\cdot\mathcal{L}\{\cos(2t)\}(s) = \mathcal{L}\{y\}(s)\cdot\frac{s}{s^2+4}.$$

Substituting this expression in (*) produces

$$\mathcal{L}\{y\}(s) + 5\mathcal{L}\{y\}(s)\cdot\frac{s}{s^2+4} = \frac{1}{s+3}.$$

Rearranging yields

$$\mathcal{L}\{y\}(s)\left[1 + \frac{5s}{s^2+4}\right] = \frac{1}{s+3}$$

or

$$\mathcal{L}\{y\}(s)\left[\frac{s^2+4+5s}{s^2+4}\right] = \frac{1}{s+3}$$

so

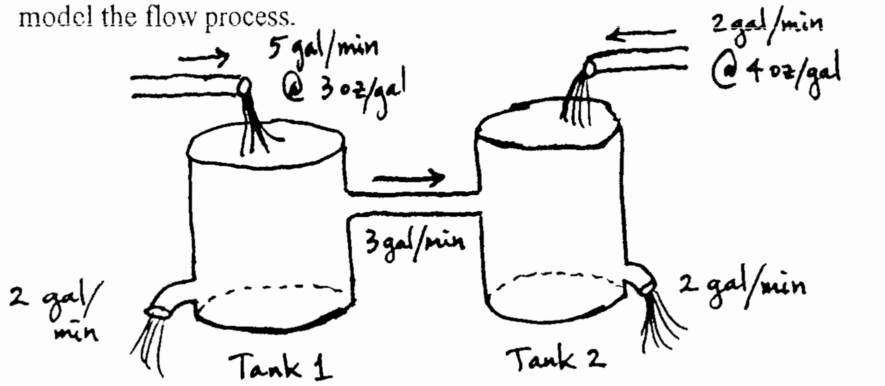
$$\mathcal{L}\{y\}(s) = \frac{s^2+4}{(s+3)(s^2+5s+4)} = \frac{s^2+4}{(s+3)(s+4)(s+1)} \overset{\text{P.F.D.}}{=} \frac{A}{s+3} + \frac{B}{s+4} + \frac{C}{s+1}.$$

Then $s^2+4 = A(s+4)(s+1) + B(s+3)(s+1) + C(s+3)(s+4)$. To find A, set $s=-3$: $(-3)^2+4 = A(1)(-2) \Rightarrow A = -13/2$. To find B, set $s=-4$: $(-4)^2+4 = B(-1)(-3)$ so $B = 20/3$. To find C, set $s=-1$: $(-1)^2+4 = C(2)(3)$ so $C = 5/6$. Therefore

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-13/2}{s+3} + \frac{20/3}{s+4} + \frac{5/6}{s+1}\right\}$$

$$\boxed{y(t) = -\frac{13}{2}e^{-3t} + \frac{20}{3}e^{-4t} + \frac{5}{6}e^{-t}}.$$

4.[18] Two very large tanks initially hold 100 gallons of pure water each. A mixture of salt and water at a concentration of 3 ounces per gallon flows into Tank 1 at a rate of 5 gallons per minute. The well-stirred mixture drains into the environment at a rate of 2 gallons per minute, and from Tank 1 into Tank 2 at a rate of 3 gallons per minute. Another mixture of salt and water at a concentration of 4 ounces per gallon flows into Tank 2 at a rate of 2 gallons per minute. The well-stirred mixture in Tank 2 drains into the environment at a rate of 2 gallons per minute. If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in ounces at time t in Tanks 1 and 2, respectively, **SET UP, BUT DO NOT SOLVE,** the differential equations and initial conditions that model the flow process.



Note that 5 gal/min of fluid is flowing into Tank 1 and 5 gal/min of fluid is flowing out of Tank 1 so the volume of fluid in Tank 1 is constant: 100 gal.
However, 5 gal/min of fluid is flowing into Tank 2 and 2 gal/min is flowing out of Tank 2 so the volume of fluid in Tank 2 increases by 3 gal each min. Thus

$$V(t) = 100 + 3t$$

is the volume of fluid (in gal.) in Tank 2 at time t (in min.)

We use the principle

$$\begin{matrix}\text{Net rate of} \\ \text{change of salt}\end{matrix} = \begin{matrix}\text{Inflow rate} \\ \text{of salt}\end{matrix} - \begin{matrix}\text{Outflow rate} \\ \text{of salt}\end{matrix}$$

and apply it to each tank.

Tank 1: $$\frac{dQ_1}{dt} = \left(\frac{5\text{ gal}}{\text{min}}\right)\left(\frac{3\text{ oz.}}{\text{gal}}\right) - \left(\frac{5\text{ gal}}{\text{min}}\right)\left(\frac{Q_1\text{ oz.}}{100\text{ gal}}\right)$$

Tank 2: $$\frac{dQ_2}{dt} = \left(\frac{2\text{ gal}}{\text{min}}\right)\left(\frac{4\text{ oz.}}{\text{gal}}\right) + \left(\frac{3\text{ gal}}{\text{min}}\right)\left(\frac{Q_1\text{ oz.}}{100\text{ gal}}\right) - \left(\frac{2\text{ gal}}{\text{min}}\right)\left(\frac{Q_2\text{ oz.}}{(100+3t)\text{ gal}}\right)$$

Simplifying and adding the initial conditions gives

$$\boxed{\begin{aligned} \frac{dQ_1}{dt} &= -\frac{1}{20}Q_1 + 15, \quad Q_1(0) = 0, \\ \frac{dQ_2}{dt} &= \frac{3}{100}Q_1 - \frac{2Q_2}{100+3t} + 8, \quad Q_2(0) = 0. \end{aligned}}$$

5.[10] Verify that the matrix function $\mathbf{X}(t)=\begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$ satisfies the differential equation

$$\mathbf{X}'=\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{X}.$$

$$X'(t) - AX(t) = \begin{bmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{bmatrix} - \begin{bmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore $X'(t) = AX(t)$; i.e. the matrix function $X(t) = \begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix}$ satisfies the differential equation $X' = AX$ where

$$A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}.$$

6.[18] Solve the initial value problem

$$x' = -5x + y, \quad x(0) = 2,$$
$$y' = -3x - y, \quad y(0) = -1.$$

Describe the behavior of the solution as $t \to \infty$.

We write the system of differential equations in vector-matrix form as

$$\vec{x}' = A\vec{x} \qquad \text{where} \qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } A = \begin{bmatrix} -5 & 1 \\ -3 & -1 \end{bmatrix}.$$

$\vec{x} = \vec{k}e^{\lambda t}$ in the DE leads to $\lambda \vec{k} = A\vec{k}$, the eigenvalue equation for the matrix A. Therefore the eigenvalues λ of A are solutions to

$$0 = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 1 \\ -3 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda+5) + 3 = \lambda^2 + 6\lambda + 8 = (\lambda+4)(\lambda+2).$$

Eigenvalues	Eigenvectors
$\lambda_1 = -4$	$\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = -2$	$\vec{k}^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Eigenvectors $\vec{k}^{(1)}$ corresponding to $\lambda_1 = -4$ satisfy

$$(A - \lambda_1 I)\vec{k}^{(1)} = \vec{0}.$$

I.e. $\begin{bmatrix} -5-(-4) & 1 \\ -3 & -1-(-4) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} -k_1 + k_2 = 0 & \therefore\ k_2 = k_1 \\ \cancel{-3k_1 + 3k_2} = 0 & \text{Redundant equation; it is 3 times the first equation.} \end{cases}$$

$\therefore \vec{k}^{(1)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Take $k_1 = 1$.

Eigenvectors $\vec{k}^{(2)}$ corresponding to $\lambda_2 = -2$ satisfy $(A - \lambda_2 I)\vec{k}^{(2)} = \vec{0}$; i.e.

$$\begin{bmatrix} -5-(-2) & 1 \\ -3 & -1-(-2) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -3k_1 + k_2 = 0 & \therefore\ k_2 = 3k_1 \\ \cancel{-3k_1 + k_2} = 0 & \text{Redundant equation.} \end{cases}$$

$\therefore \vec{k}^{(2)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ 3k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Take $k_1 = 1$.

The general solution of $\vec{x}' = A\vec{x}$ is $\vec{x}(t) = c_1\vec{k}^{(1)}e^{\lambda_1 t} + c_2\vec{k}^{(2)}e^{\lambda_2 t} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-4t} + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-2t}$.

We want to choose the arbitrary constants c_1 and c_2 so the initial condition is met:

$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. By inspection, $c_1 = \frac{7}{2}$ and $c_2 = \frac{-3}{2}$. Thus the solution to the I.V.P. is $\boxed{\vec{x}(t) = \frac{7}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-4t} - \frac{3}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-2t}}$. Equivalently

$\boxed{x(t) = \frac{7}{2}e^{-4t} - \frac{3}{2}e^{-2t} \text{ and } y(t) = \frac{7}{2}e^{-4t} - \frac{9}{2}e^{-2t}}$. Clearly $\boxed{\vec{x}(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ as } t \to \infty}$.

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\dfrac{1}{s-a}$
2. t^n	$\dfrac{n!}{s^{n+1}}, \quad n = 0,1,2,3....$
3. $\sin(bt)$	$\dfrac{b}{s^2+b^2}$
4. $\cos(bt)$	$\dfrac{s}{s^2+b^2}$
5. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
6. $e^{ct} f(t)$	$F(s-c)$
7. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
8. $(f * g)(t)$	$F(s) G(s)$
9. $\delta(t-c)$	e^{-cs}

2012 Spr. Semester, Math 204 Hour Exam III, Master List

100	||||	59	卌	19	
99		58	|||	18	
98	卌	57	||||	17	
97	卌 ||	56	卌 |	16	
96	卌	55	|	15	
95	|||	54	||||	14	
94	卌 ||	53	||	13	
93	||||	52	||||	12	
92	卌 |	51	||	11	
91	卌 ||	50	|	10	
90	卌 ||	49	||	9	
89	卌 卌 卌 |	48	||	8	
88	卌 ||||	47		7	
87	卌 卌	46		6	
86	||||	45	|	5	
85	||||	44	||||	4	
84	卌 |	43		3	
83	卌 |||	42		2	
82	卌 |||	41		1	
81	卌 ||	40		0	
80	||||	39	||		
79	卌 ||||	38	||		
78	卌 |	37			
77	卌 |||	36			
76	卌 |||	35			
75	卌 |||	34	|		
74	卌 |	33			
73	卌 ||||	32			
72	|||	31			
71	||||	30			
70	||||	29			
69	||||	28	|		
68	卌 ||	27			
67	卌 |||	26	|		
66	卌	25			
65	卌 ||||	24			
64	||	23	|		
63	||||	22			
62	||||	21			
61	||||	20	|		
60	||||				

55 As

76 Bs

65 Cs

51 Ds

50 Fs

Number taking exam: 297
Median: 77
Mean: 75.2
Standard Deviation: 15.7

Number receiving A's: 55 18.5%
Number receiving B's: 76 25.6
Number receiving C's: 65 21.9
Number receiving D's: 51 17.2
Number receiving F's: 50 16.8

2012 Spring Semester, Math 204 Hour Exam III
Instructor Grow, Section A

Score	Tally		Score	Tally		Score
100	I		59			19
99			58			18
98	I		57			17
97	I		56			16
96			55			15
95			54	I		14
94	II	8 As	53			13
93	I		52			12
92			51			11
91	I		50		1 F	10
90	I		49			9
89	I		48			8
88	I		47			7
87			46			6
86	II		45			5
85	I		44			4
84		9 Bs	43			3
83	II		42			2
82	I		41			1
81	I		40			0
80			39			
79			38			
78			37			
77	I		36			
76			35			
75	II	6 Cs	34			
74			33			
73	II		32			
72			31			
71			30			
70	I		29			
69	II		28			
68	I		27			
67	I		26			
66			25			
65	II	6 Ds	24			
64			23			
63			22			
62			21			
61			20			
60						

Number taking exam: 30
Median: 82.5
Mean: 80.7
Standard Deviation: 11.7

Number receiving A's:	8	26.7 %
Number receiving B's:	9	30.0
Number receiving C's:	6	20.0
Number receiving D's:	6	20.0
Number receiving F's:	1	3.3