**Mathematics 204**

**Fall 2011**

**Final Exam**

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. The final exam consists of this cover page, 10 pages of problems containing 10 numbered problems, and a short table of Laplace transform formulas.
5. Once the exam begins, you will have 120 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, work must be shown on integration, partial fraction, and matrix computations.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 200.

| **problem** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | **Sum** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **points earned** | | | | | | | | | | | |
| **maximum points** | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 200 |

1.[20] Find the general solution of $y' = 2 - \dfrac{3y}{50+t}$ on the interval $t > -50$.

$$y' + \left(\frac{3}{50+t}\right) y = 2 \qquad \text{(1st order, linear)}$$

Integrating factor: $e^{\int p(t)dt} = e^{\int \frac{3}{50+t}dt} = e^{3\ln(50+t) + \overset{0}{\cancel{c}}} = e^{\ln(50+t)^3} = (50+t)^3$

$$(50+t)^3\left[y' + \frac{3}{50+t}y\right] = 2(50+t)^3$$

$$\underbrace{(50+t)^3 y' + 3(50+t)^2 y}_{\text{Exact!}} = 2(50+t)^3$$

$$\frac{d}{dt}\left[(50+t)^3 y\right] = 2(50+t)^3$$

$$(50+t)^3 y = \int 2(50+t)^3 dt = \frac{1}{2}(50+t)^4 + c$$

$$\therefore \quad \boxed{y(t) = \frac{1}{2}(50+t) + \frac{c}{(50+t)^3}}$$

where c is an arbitrary constant.

1st order, variables separable

2.[20] Find the explicit solution of the initial value problem $y' = \frac{3t^2 - 1}{2 + 2y}$, $y(1) = -2$.

$$\frac{dy}{dt} = \frac{3t^2-1}{2+2y}$$

$$(2+2y)dy = (3t^2-1)dt$$

$$2y + y^2 = \int (2+2y)dy = \int (3t^2-1)dt = t^3 - t + \tilde{c}$$

$$\underbrace{1 + 2y + y^2}_{\text{Completed square}} = t^3 - t + c \qquad (c = \tilde{c} + 1)$$

$$(y+1)^2 = t^3 - t + c$$

$$y + 1 = \pm\sqrt{t^3 - t + c}$$

$$y(t) = -1 \pm \sqrt{t^3 - t + c}$$

Choose $-$ sign in order to satisfy the initial condition $y(1) = -2$.

$$-2 = y(1) = -1 - \sqrt{1^3 - 1 + c}$$

$$\Rightarrow c = 1.$$

$$\therefore \boxed{y(t) = -1 - \sqrt{t^3 - t + 1}}$$

| $t$ | Volume in Tank |
|---|---|
| 0 | 100 gal |
| 1 | 102 |
| 2 | 104 |
| ⋮ | ⋮ |
| $t$ | $100+2t$ |

$Q(0) = 0$

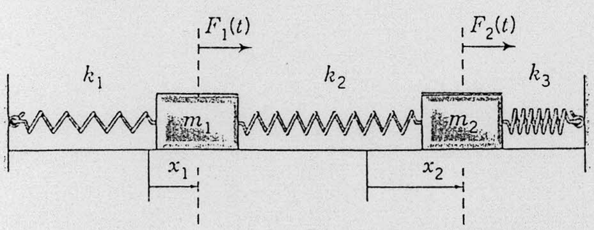
3. (a)[10] A 200 gallon tank initially contains 100 gallons of pure water. Water containing 1/4 pound of salt per gallon enters the tank at a rate of 8 gallons per minute and the well-stirred mixture exits the tank at 6 gallons per minute. Set up, **BUT DO NOT SOLVE**, an initial value problem that models the number of pounds $Q(t)$ of salt in the tank at time $t$ minutes over the time interval $0 \le t \le 50$.

Net Rate = Rate In − Rate Out

$$\frac{dQ}{dt} = \left(\frac{1/4 \text{ lb}}{\cancel{\text{gal}}}\right)\left(\frac{8 \cancel{\text{gal}}}{\text{min}}\right) - \left(\frac{Q(t) \text{ lb}}{(100+2t)\cancel{\text{gal}}}\right)\left(\frac{6 \cancel{\text{gal}}}{\text{min}}\right)$$

$$\boxed{\frac{dQ}{dt} = 2 - \frac{3Q}{50+t}, \quad Q(0) = 0}$$

(b)[10] Consider the coupled vibrating system in the figure below. Two bodies with masses $m_1$ and $m_2$, respectively, move on a frictionless surface under the influence of horizontal external forces $F_1(t)$ and $F_2(t)$, and they are also constrained by three springs whose constants are $k_1$, $k_2$, and $k_3$, respectively. If the horizontal displacements at time $t$ of the bodies from their static equilibrium positions are $x_1(t)$ and $x_2(t)$, respectively, use Newton's second law, $F = ma$, to help write a system of differential equations governing the motion of the bodies. (DO NOT SOLVE THIS SYSTEM!)




Convention: Displacements and forces to the right are positive.

Apply Newton's second law to Body 1:

$$m_1 x_1'' = F_{\text{spring 1}} + F_{\text{spring 2}} + F_1(t)$$
$$= -k_1 x_1 + k_2(x_2 - x_1) + F_1(t)$$

Apply Newton's second law to Body 2:

$$m_2 x_2'' = F_{\text{spring 2}} + F_{\text{spring 3}} + F_2(t)$$
$$= -k_2(x_2 - x_1) - k_3 x_2 + F_2(t)$$

Therefore

$$\boxed{\begin{aligned} m_1 x_1'' &= -(k_1+k_2)x_1 + k_2 x_2 + F_1(t) \\ m_2 x_2'' &= k_2 x_1 - (k_2+k_3)x_2 + F_2(t) \end{aligned}}$$

governs the motion of the bodies.

$$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = \frac{1}{t} \qquad \text{so} \qquad g(t) = \frac{1}{t}.$$

4.[20] Solve $t^2y'' - ty' + y = t$ on the interval $t > 0$. (Nonhomogeneous, second-order Euler eqn.)

$y = t^m$ in $t^2y'' - ty' + y = 0$ leads to $m(m-1) - m + 1 = 0 \Rightarrow$

$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$ (multiplicity 2).

Therefore $y_c(t) = c_1 t + c_2 t\ln(t)$ is the general solution of $t^2y'' - ty' + y = 0$ on $t > 0$.

To find a particular solution of the nonhomogeneous DE, we use variation of parameters:

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where $y_1(t) = t$, $y_2(t) = t\ln(t)$,

$$u_1(t) = \int \frac{-y_2(t)g(t)}{W(t)}\,dt, \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W(t)}\,dt$$

$g(t) = \frac{1}{t}$, and $W(t) = \begin{vmatrix} t & t\ln(t) \\ 1 & 1 + \ln(t) \end{vmatrix} = t$. Thus

$$u_1(t) = \int \frac{-t\ln(t)\cdot\frac{1}{t}}{t}\,dt = -\int \underbrace{\ln(t)}_{v}\,\underbrace{\tfrac{1}{t}\,dt}_{dv} = -\frac{1}{2}(\ln(t))^2 + \cancel{c}^{\,0}$$

$$u_2(t) = \int \frac{t\cdot\frac{1}{t}}{t}\,dt = \int \frac{1}{t}\,dt = \ln(t) + \cancel{c}^{\,0}.$$

Consequently, $y_p(t) = -\frac{1}{2}\ln^2(t)\cdot t + \ln(t)\cdot t\ln(t) = \frac{t}{2}\ln^2(t)$

The general solution is $y = y_c + y_p$

$$\boxed{y(t) = c_1 t + c_2 t\ln(t) + \frac{t}{2}\ln^2(t)}$$

where $c_1$ and $c_2$ are arbitrary constants.

5.[20] Find the general solution of the differential equation $y^{(4)} - y''' + y'' = e^{2t}$.

( Fourth order, constant coefficent, linear, nonhomogeneous DE )

$y = e^{rt}$ in $y^{(4)} - y''' + y'' = 0$ leads to $r^4 - r^3 + r^2 = 0 \Rightarrow r^2(r^2 - r + 1) = 0$

$\Rightarrow r = 0$ (multiplicity 2) or $r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The general solution of $y^{(4)} - y''' + y'' = 0$ is $y_c(t) = c_1 + c_2 t + c_3 e^{t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) + c_4 e^{t/2} \sin\left(\frac{\sqrt{3}t}{2}\right)$.

We use undetermined coefficients to find a particular solution of the nonhomogeneous DE. We use $y_p(t) = Ae^{2t}$ as a trial particular solution where A is a constant to be determined. Then $y_p' = 2Ae^{2t}$, $y_p'' = 4Ae^{2t}$, $y_p''' = 8Ae^{2t}$, $y_p^{(4)} = 16Ae^{2t}$. Then

$$y_p^{(4)} - y_p''' + y_p'' = e^{2t}$$

leads to

$$16A\cancel{e^{2t}} - 8A\cancel{e^{2t}} + 4A\cancel{e^{2t}} = \cancel{e^{2t}}^{\,1}$$

or

$$12A = 1 \quad \Rightarrow \quad A = 1/12.$$

Therefore $y_p(t) = \frac{1}{12}e^{2t}$. The general solution of the nonhomogeneous DE is

$y = y_c + y_p$ or

$$\boxed{y(t) = c_1 + c_2 t + e^{t/2}\left(c_3 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_4 \sin\left(\frac{\sqrt{3}}{2}t\right)\right) + \frac{1}{12}e^{2t}}$$

where $c_1, c_2, c_3,$ and $c_4$ are arbitrary constants.

6.[20] Find the solution of the initial value problem

$$y'' + y = \delta(t-\pi)\cos(t), \quad y(0)=0,\ y'(0)=1,$$

and sketch the graph of the solution on the interval $0 \le t \le 2\pi$.

We use the method of Laplace transforms.

$$\mathcal{L}\{y''+y\}(s) = \mathcal{L}\{\delta(t-\pi)\cos(t)\}(s)$$

$$s^2\mathcal{L}\{y\}(s) - s\overset{0}{\cancel{y(0)}} - \overset{1}{\cancel{y'(0)}} + \mathcal{L}\{y\}(s) = \int_0^\infty \delta(t-\pi)\cos(t)e^{-st}\,dt = \cos(\pi)e^{-s\pi}$$

This is by the "sifting" property of the Dirac delta.

$$\therefore \quad (s^2+1)\mathcal{L}\{y\}(s) = 1 - e^{-s\pi}$$

$$\mathcal{L}\{y\}(s) = \frac{1}{s^2+1} - \frac{e^{-s\pi}}{s^2+1}.$$

Taking the inverse Laplace transform of both sides gives

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}\right\}$$

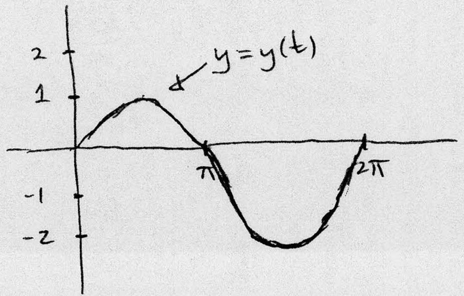
$$= \sin(t) - f(t-\pi)u_\pi(t)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$. Therefore $y(t) = \sin(t) - \sin(t-\pi)u_\pi(t)$.

But $\sin(t-\pi) = \sin(t)\cos(\pi) - \cos(t)\sin(\pi) = -\sin(t)$ so $\boxed{y(t) = \sin(t) + \sin(t)u_\pi(t)}$

That is,

$$y(t) = \begin{cases} \sin(t) & \text{if } 0 \le t < \pi, \\ 2\sin(t) & \text{if } \pi \le t < \infty. \end{cases}$$



7.[20] Solve the integral equation $y(t)+\int_0^t e^{\tau} y(t-\tau)\,d\tau=\cos(t)$.

We use the method of Laplace transforms. We first rewrite the equation using the convolution product:

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Therefore, the original equation is

$$y(t)+(f*y)(t)=\cos(t)$$

where $f(t)=e^t$. Taking Laplace transforms and using $\mathcal{L}\{f*g\}(s)=F(s)G(s)$ gives

$$\mathcal{L}\{y+(f*y)\}(s)=\mathcal{L}\{\cos(t)\}(s)$$

$$\mathcal{L}\{y\}(s)+\mathcal{L}\{f\}(s)\,\mathcal{L}\{y\}(s)=\frac{s}{s^2+1}$$

$$\mathcal{L}\{y\}(s)+\frac{1}{s-1}\cdot\mathcal{L}\{y\}(s)=\frac{s}{s^2+1}$$

$$\left(1+\frac{1}{s-1}\right)\mathcal{L}\{y\}(s)=\frac{s}{s^2+1}$$

$$\left(\frac{s-\cancel{1}+\cancel{1}}{s-1}\right)\mathcal{L}\{y\}(s)=\frac{s}{s^2+1}$$

$$\mathcal{L}\{y\}(s)=\frac{s-1}{\cancel{s}}\cdot\frac{\cancel{s}}{s^2+1}=\frac{s}{s^2+1}-\frac{1}{s^2+1}$$

Taking the inverse transform yields

$$y(t)=\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}-\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t)=\cos(t)-\sin(t)}\,.$$

8.[20] Find the general solution of the system $\mathbf{x}' = \overbrace{\begin{pmatrix} 2 & -4 \\ 4 & -6 \end{pmatrix}}^{A}\mathbf{x}$ and describe the behavior of solutions as $t \to \infty$.

$\vec{x} = \vec{k}e^{\lambda t}$ in $\vec{x}' = A\vec{x}$ leads to $\lambda\vec{k} = A\vec{k}$. (This is the eigenvalue equation for A.)

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -4 \\ 4 & -6-\lambda \end{vmatrix} = (\lambda+6)(\lambda-2)+16 = \lambda^2+4\lambda+4 = (\lambda+2)^2.$$

| Eigenvalues | Eigenvectors |
|---|---|
| $\lambda_1 = -2$ | $\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ |
| $\lambda_2 = -2$ | (There is no second l.i. eigenvector) |

An eigenvector $\vec{k}$ of A corresponding to an eigenvalue $\lambda$ satisfies $(A - \lambda I)\vec{k} = \vec{0}$. When $\lambda = -2$, this becomes

$$\begin{bmatrix} 2-(-2) & -4 \\ 4 & -6-(-2) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 4k_1 - 4k_2 = 0 \\ \cancel{4k_1 - 4k_2 = 0} \text{ Redundant} \end{cases}$$

$$\Rightarrow k_2 = k_1 \text{ so } \vec{k}^{(1)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}k_1.$$

$\vec{x} = \vec{k}te^{\lambda t} + \vec{\ell}e^{\lambda t}$ in $\vec{x}' = A\vec{x}$ leads to $\begin{cases} (A - \lambda I)\vec{k} = 0, \\ (A - \lambda I)\vec{\ell} = \vec{k}. \end{cases}$

We know that $\lambda = -2$ and $\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ satisfy the first equation. Therefore we need to solve $(A - (-2)I)\vec{\ell} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, i.e. $\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$

$$\begin{cases} 4\ell_1 - 4\ell_2 = 1 \Rightarrow \ell_2 = \ell_1 - \frac{1}{4}. \\ \cancel{4\ell_1 - 4\ell_2 = 1} \text{ Redundant} \end{cases}$$

Thus $\vec{\ell} = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = \begin{bmatrix} \ell_1 \\ \ell_1 - 1/4 \end{bmatrix} = \ell_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}$.

We take $\ell_1 = 0$, and obtain $\vec{\ell} = \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}$. Therefore

$$\vec{x}^{(1)}(t) = \vec{k}^{(1)}e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} \quad \text{and} \quad \vec{x}^{(2)}(t) = t\vec{k}e^{\lambda_1 t} + \vec{\ell}e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}te^{-2t} + \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}e^{-2t}$$

are l.i. solutions to $\vec{x}' = A\vec{x}$. The general solution is

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}te^{-2t} + \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}e^{-2t}\right)}$$

where $c_1$ and $c_2$ are arbitrary constants

Since $e^{-2t} \to 0$ and $te^{-2t} \to 0$ as $t \to \infty$, we have $\boxed{\vec{x}(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ as } t \to \infty.}$

9.[20] Given that $\mathbf{x}^{(1)}(t)=\begin{pmatrix}1\\-4\end{pmatrix}e^{-3t}$ and $\mathbf{x}^{(2)}(t)=\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$ form a fundamental set of solutions for the homogeneous system $\mathbf{x}'=\underbrace{\begin{pmatrix}1 & 1\\4 & -2\end{pmatrix}}_{A}\mathbf{x}$, solve the nonhomogeneous initial value problem

$$\mathbf{x}'=\begin{pmatrix}1 & 1\\4 & -2\end{pmatrix}\mathbf{x}+\begin{pmatrix}te^{2t}\\te^{2t}\end{pmatrix},\quad \mathbf{x}(0)=\begin{pmatrix}0\\5\end{pmatrix}.$$

$$\Psi(t)=\left[\vec{x}^{(1)}(t),\ \vec{x}^{(2)}(t)\right]=\begin{bmatrix}e^{-3t} & e^{2t}\\-4e^{-3t} & e^{2t}\end{bmatrix}$$ is a fundamental matrix for $\vec{x}'=A\vec{x}$.

A particular solution for the nonhomogeneous system can be obtained using variation of parameters: $\vec{x}_p(t)=\Psi(t)\int_{t_0}^{t}\Psi^{-1}(s)\vec{g}(s)\,ds$.

$$\Psi^{-1}(s)=\frac{1}{\det\Psi(s)}\begin{bmatrix}\psi_{22} & -\psi_{12}\\-\psi_{21} & \psi_{11}\end{bmatrix}=\frac{1}{5e^{-s}}\begin{bmatrix}e^{2s} & -e^{2s}\\4e^{-3s} & e^{-3s}\end{bmatrix}=\frac{1}{5}\begin{bmatrix}e^{3s} & -e^{3s}\\4e^{-2s} & e^{-2s}\end{bmatrix}$$

$$\Psi^{-1}(s)\vec{g}(s)=\frac{1}{5}\begin{bmatrix}e^{3s} & -e^{3s}\\4e^{-2s} & e^{-2s}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}se^{2s}=\frac{1}{5}\begin{bmatrix}0\\5e^{-2s}\end{bmatrix}se^{2s}=\begin{bmatrix}0\\s\end{bmatrix}.$$

$$\int_0^t\Psi^{-1}(s)\vec{g}(s)\,ds=\int_0^t\begin{bmatrix}0\\s\end{bmatrix}ds=\begin{bmatrix}0\\t^2/2\end{bmatrix}.$$

$$\therefore\ \vec{x}_p(t)=\Psi(t)\begin{bmatrix}0\\t^2/2\end{bmatrix}=\frac{t^2}{2}\begin{bmatrix}e^{2t}\\e^{2t}\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}\frac{t^2}{2}e^{2t}.$$

The general solution of the nonhomogeneous system is $\vec{x}(t)=\vec{x}_c(t)+\vec{x}_p(t)$ or

$$\vec{x}(t)=\Psi(t)\vec{c}+\begin{bmatrix}1\\1\end{bmatrix}\frac{t^2}{2}e^{2t}=\begin{bmatrix}e^{-3t} & e^{2t}\\-4e^{-3t} & e^{2t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\frac{t^2}{2}e^{2t}.$$

$$\therefore\ \begin{bmatrix}0\\5\end{bmatrix}=\vec{x}(0)=\begin{bmatrix}1 & 1\\-4 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}0\ \Rightarrow\ \begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1 & 1\\-4 & 1\end{bmatrix}^{-1}\begin{bmatrix}0\\5\end{bmatrix}=\frac{1}{5}\begin{bmatrix}1 & -1\\4 & 1\end{bmatrix}\begin{bmatrix}0\\5\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}.$$

The solution of the I.V.P. is

$$\boxed{\vec{x}(t)=\begin{bmatrix}e^{-3t} & e^{2t}\\-4e^{-3t} & e^{2t}\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\frac{t^2}{2}e^{2t}}$$

or $$\boxed{\vec{x}(t)=-\begin{bmatrix}1\\-4\end{bmatrix}e^{-3t}+\begin{bmatrix}1\\1\end{bmatrix}e^{2t}+\begin{bmatrix}1\\1\end{bmatrix}\frac{t^2}{2}e^{2t}}.$$

10.[20] Solve the initial value problem

Method I (matrix method)

$$x''=-2x+y,\quad x(0)=1,\ x'(0)=0,$$
$$y''=2x-3y,\quad y(0)=-2,\ y'(0)=0.$$

Write $\vec{x}=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$. Then the IVP can be expressed as $\vec{x}''=A\vec{x}$, $\vec{x}(0)=\begin{bmatrix}1\\-2\end{bmatrix}$, $\vec{x}'(0)=\begin{bmatrix}0\\0\end{bmatrix}$ where $A=\begin{bmatrix}-2&1\\2&-3\end{bmatrix}$. Thus $\vec{x}=\vec{k}e^{rt}$ in $\vec{x}''=A\vec{x}$ leads to $r^2\vec{k}=A\vec{k}$. This is the eigenvalue equation for A with $\lambda=r^2$.

$$0=\det(A-\lambda I)=\begin{vmatrix}-2-\lambda&1\\2&-3-\lambda\end{vmatrix}=(\lambda+3)(\lambda+2)-2=\lambda^2+5\lambda+4=(\lambda+1)(\lambda+4).$$

| Eigenvalues | Eigenvectors |
|---|---|
| $\lambda_1=-1$ | $\vec{k}^{(1)}=\begin{bmatrix}1\\1\end{bmatrix}$ |
| $\lambda_2=-4$ | $\vec{k}^{(2)}=\begin{bmatrix}1\\-2\end{bmatrix}$ |

Eigenvectors $\vec{k}$ satisfy $(A-\lambda I)\vec{k}=\vec{0}$.

$\lambda_1=-1$: $\begin{bmatrix}-2-(-1)&1\\2&-3-(-1)\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\Rightarrow\begin{cases}-k_1+k_2=0\\ \cancel{2k_1-2k_2=0}\ \text{Redund.}\end{cases}$

$\vec{k}^{(1)}=\begin{bmatrix}k_1\\k_2\end{bmatrix}=\begin{bmatrix}k_1\\k_1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}k_1$.

$\lambda_2=-4$: $\begin{bmatrix}-2-(-4)&1\\2&-3-(-4)\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\Rightarrow\begin{cases}2k_1+k_2=0\\2k_1+k_2=0\end{cases}\Rightarrow k_2=-2k_1$ so $\vec{k}^{(2)}=\begin{bmatrix}k_1\\k_2\end{bmatrix}=\begin{bmatrix}k_1\\-2k_1\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}$

If $r_1^2=\lambda_1=-1$ then $r_1=\pm i$. If $r_2^2=\lambda_2=-4$ then $r_2=\pm 2i$. Therefore $\tilde{\vec{x}}^{(1)}=\vec{k}^{(1)}e^{r_1t}=\begin{bmatrix}1\\1\end{bmatrix}e^{it}=\begin{bmatrix}1\\1\end{bmatrix}\cos(t)+i\begin{bmatrix}1\\1\end{bmatrix}\sin(t)$ and $\tilde{\vec{x}}^{(2)}=\vec{k}^{(2)}e^{r_2t}=\begin{bmatrix}1\\-2\end{bmatrix}e^{2it}=\begin{bmatrix}1\\-2\end{bmatrix}\cos(2t)+i\begin{bmatrix}1\\-2\end{bmatrix}\sin(2t)$ are complex solutions to $\vec{x}''=A\vec{x}$. Consequently, real solutions to $\vec{x}''=A\vec{x}$ are $\vec{x}^{(1)}(t)=Re(\tilde{\vec{x}}^{(1)})=\begin{bmatrix}1\\1\end{bmatrix}\cos(t)$, $\vec{x}^{(2)}(t)=Im(\tilde{\vec{x}}^{(1)})=\begin{bmatrix}1\\1\end{bmatrix}\sin(t)$, $\vec{x}^{(3)}(t)=Re(\tilde{\vec{x}}^{(2)})=\begin{bmatrix}1\\-2\end{bmatrix}$, $\vec{x}^{(4)}(t)=Im(\tilde{\vec{x}}^{(2)})=\begin{bmatrix}1\\-2\end{bmatrix}\sin(2t)$ so $\vec{x}(t)=c_1\begin{bmatrix}1\\1\end{bmatrix}\cos(t)+c_2\begin{bmatrix}1\\1\end{bmatrix}\sin(t)+c_3\begin{bmatrix}1\\-2\end{bmatrix}\cos(2t)+c_4\begin{bmatrix}1\\-2\end{bmatrix}\sin(2t)$ is the general solution to $\vec{x}''=A\vec{x}$. Note that $\vec{x}'(t)=-c_1\begin{bmatrix}1\\1\end{bmatrix}\sin(t)+c_2\begin{bmatrix}1\\1\end{bmatrix}\cos(t)-2c_3\begin{bmatrix}1\\-2\end{bmatrix}\sin(2t)+2c_4\begin{bmatrix}1\\-2\end{bmatrix}\cos(2t)$.

$$\therefore\ \begin{bmatrix}1\\-2\end{bmatrix}=\vec{x}(0)=c_1\begin{bmatrix}1\\1\end{bmatrix}+c_3\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}1&1\\1&-2\end{bmatrix}\begin{bmatrix}c_1\\c_3\end{bmatrix}\Rightarrow\begin{bmatrix}c_1\\c_3\end{bmatrix}=\begin{bmatrix}1&1\\1&-2\end{bmatrix}^{-1}\begin{bmatrix}1\\-2\end{bmatrix}=\frac{1}{-3}\begin{bmatrix}-2&-1\\-1&1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix}=\vec{x}'(0)=c_2\begin{bmatrix}1\\1\end{bmatrix}+2c_4\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}1&2\\1&-4\end{bmatrix}\begin{bmatrix}c_2\\c_4\end{bmatrix}\Rightarrow\begin{bmatrix}c_2\\c_4\end{bmatrix}=\begin{bmatrix}1&2\\1&-4\end{bmatrix}^{-1}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Thus $\vec{x}(t)=\begin{bmatrix}1\\-2\end{bmatrix}\cos(2t)$ or equivalently $\boxed{\begin{aligned}x(t)&=\cos(2t)\\y(t)&=-2\cos(2t)\end{aligned}}$ solves the I.V.P.

## Method II (reduction to a single higher order DE)

Solving the first DE in the system for $y$ gives $y = x'' + 2x$. Substituting this expression for $y$ in the second DE of the system yields $(x''+2x)'' = 2x - 3(x''+2x)$, or equivalently

$$x^{(4)} + 2x'' = 2x - 3x'' - 6x$$

$$(*) \qquad x^{(4)} + 5x'' + 4x = 0 .$$

$x = e^{rt}$ leads to $r^4 + 5r^2 + 4 = 0 \Rightarrow (r^2+4)(r^2+1) = 0 \Rightarrow r = \pm 2i,\ r = \pm i$.

Therefore $x(t) = c_1\cos(2t) + c_2\sin(2t) + c_3\cos(t) + c_4\sin(t)$ is the general solution of $(*)$.

We have $x'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) - c_3\sin(t) + c_4\cos(t)$,

$$x''(t) = -4c_1\cos(2t) - 4c_2\sin(2t) - c_3\cos(t) - c_4\sin(t),$$

$$x'''(t) = +8c_1\sin(2t) - 8c_2\cos(2t) + c_3\sin(t) - c_4\cos(t).$$

We apply the initial conditions to obtain

$(\dagger) \qquad 1 = x(0) = c_1 + c_3 \qquad$ and $\qquad 0 = x'(0) = 2c_2 + c_4$.

By the system's first DE, $x'' = -2x + y$ and $x''' = -2x' + y'$, so applying the initial conditions yields $x''(0) = -2x(0) + y(0) = -2\cdot 1 + -2 = -4$ and $x'''(0) = -2x'(0) + y'(0) = 0$. Therefore

$(\dagger\dagger) \qquad -4 = x''(0) = -4c_1 - c_3 \qquad$ and $\qquad 0 = x'''(0) = -8c_2 - c_4$.

Adding the equations of $(\dagger)$ to $(\dagger\dagger)$ yields $-3 = -3c_1$ and $0 = -6c_2$ so $c_1 = 1$ and $c_2 = 0$. Substituting these values into the equations of $(\dagger)$ gives $c_3 = 0$ and $c_4 = 0$. Therefore

$\boxed{x(t) = \cos(2t)}$. Substituting in $y = x'' + 2x$ yields $y(t) = -4\cos(2t) + 2\cos(2t)$ or

$\boxed{y(t) = -2\cos(2t)}$.

Method III: (Laplace transforms)

We take the Laplace transforms of the two DEs and apply the initial conditions.

$$\begin{cases} \mathcal{L}\{x''\}(s) = \mathcal{L}\{-2x+y\}(s) \\ \mathcal{L}\{y''\}(s) = \mathcal{L}\{2x-3y\}(s) \end{cases}$$

$$\Rightarrow \begin{cases} s^2\mathcal{L}\{x\}(s) - s\overbrace{x(0)}^{1} - \overbrace{x'(0)}^{0} = -2\mathcal{L}\{x\}(s) + \mathcal{L}\{y\}(s) \\ s^2\mathcal{L}\{y\}(s) - s\underbrace{y(0)}_{-2} - \underbrace{y'(0)}_{0} = 2\mathcal{L}\{x\}(s) - 3\mathcal{L}\{y\}(s) \end{cases}$$

$$\Rightarrow \begin{cases} (s^2+2)\mathcal{L}\{x\}(s) - \mathcal{L}\{y\}(s) = s \\ -2\mathcal{L}\{x\}(s) + (s^2+3)\mathcal{L}\{y\}(s) = -2s \end{cases}$$

We apply Cramer's rule to solve this system for $\mathcal{L}\{x\}(s)$:

$$\mathcal{L}\{x\}(s) = \frac{\begin{vmatrix} s & -1 \\ -2s & s^2+3 \end{vmatrix}}{\begin{vmatrix} s^2+2 & -1 \\ -2 & s^2+3 \end{vmatrix}} = \frac{s^3+s}{s^4+5s^2+4} = \frac{s\cancel{(s^2+1)}}{(s^2+4)\cancel{(s^2+1)}}.$$

Therefore $\boxed{x(t) =} \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \boxed{\cos(2t)}$. As in Method II, substituting for $x$ in $y = x'' + 2x$ yields $\boxed{y(t) = -2\cos(2t)}$.

# A SHORT TABLE OF LAPLACE TRANSFORMS

| $f(t)$ | $\mathcal{L}\{f(t)\} = F(s)$ |
|---|---|
| 1. $e^{at}$ | $\dfrac{1}{s-a}$ |
| 2. $t^n$ | $\dfrac{n!}{s^{n+1}}$, $\quad n = 0,1,2,3....$ |
| 3. $\sin(bt)$ | $\dfrac{b}{s^2+b^2}$ |
| 4. $\cos(bt)$ | $\dfrac{s}{s^2+b^2}$ |
| 5. $(f * g)(t)$ | $F(s)G(s)$ |
| 6. $f^{(n)}(t)$ | $s^n F(s) - s^{n-1} f(0) - ... - f^{(n-1)}(0)$ |
| 7. $e^{ct} f(t)$ | $F(s-c)$ |
| 8. $u_c(t) f(t-c)$ | $e^{-cs} F(s)$ |

**Math 204 Final Exam "Master List" Scorecard, 2011 Fall Semester**

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21 A's ✓

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136 C's ✓

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69 F's ✓

(16.7%)

Number taking final: 414 ✓
Median: 148
Mean: 142.9
Standard Deviation: 29.2