Mathematics 204

Fall 2012

Final Exam

Your Printed Name: Dr. Grow

Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. The final exam consists of this cover page, 10 pages of problems containing 10 numbered problems, and a short table of Laplace transform formulas.
5. Once the exam begins, you will have 120 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, work must be shown on integration, partial fraction, and matrix computations.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 200.

problem	1	2	3	4	5	6	7	8	9	10	**Sum**
points earned											
maximum points	14	20	21	21	20	21	21	21	21	20	200

1.[14] Determine the longest interval in which the initial value problem

$$ty'' + y' = 1 + \frac{1}{(t-2)^2}, \quad y(1) = 0,\ y'(1) = 3,$$

is guaranteed to have a unique twice differentiable solution. Do not attempt to find the solution.

We first normalize the DE so we can apply the existence-uniqueness theorem for second-order linear initial value problems:

$$y'' + \frac{1}{t}y' = \frac{1}{t} + \frac{1}{t(t-2)^2}, \quad y(1) = 0,\ y'(1) = 3.$$

This is of the form $y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_1$ where $t_0 = 1$

and $p(t) = \frac{1}{t}$, $q(t) = 0$, and $g(t) = \frac{1}{t} + \frac{1}{t(t-2)^2}$.

Since p, q, and g are continuous functions on the interval $I = (0,2)$ containing the point $t_0 = 1$, the existence-uniqueness theorem guarantees that the IVP will have a unique twice-differentiable solution throughout $\boxed{I = (0,2).}$ This is the longest interval since p is discontinuous at 0 and g is discontinuous at 0 and 2 so 0 and 2 cannot belong to the interval.

2.[20] Find the general solution of $y' = \frac{t - ty}{t^2 + 1} = \frac{t(1-y)}{t^2+1}$ <u>Method I</u>: Separation of Variables

Since the DE is of the form $y' = g(t)h(y)$, it is (first-order) separable. Writing $y' = \frac{dy}{dt}$, the DE is equivalent to $\frac{dy}{1-y} = \frac{t\,dt}{t^2+1}$, assuming that $y \neq 1$. Integrating both sides yields

$$c - \ln|1-y| = \int \frac{dy}{1-y} = \int \frac{t\,dt}{1+t^2} = \frac{1}{2}\int \frac{d(1+t^2)}{1+t^2} = \frac{1}{2}\ln(1+t^2).$$

Rearranging gives

$$c = \ln|1-y| + \ln\sqrt{1+t^2} = \ln\left(|1-y|\sqrt{1+t^2}\right)$$

and exponentiating both sides yields the implicit solution

$$K = |1-y|\sqrt{t^2+1}$$

where $K = e^c$ is an abitrary positive constant. Rearranging again,

$$\pm\frac{K}{\sqrt{t^2+1}} = 1 - y,$$

which yields the explicit solution

$$\boxed{y(t) = 1 + \frac{b}{\sqrt{t^2+1}}}$$

where b is an arbitrary constant. Note that with $b=0$ this recovers the solution $y(t) = 1$ that we lost when we divided through by $1-y$ in the initial steps of the solution.

<u>Method II</u>: Linear First-Order Equation

We can rewrite the original DE in the form

(cont.)

$$y' + \frac{t}{t^2+1} y = \frac{t}{t^2+1} .$$

An integrating factor is

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{t}{t^2+1} dt} = e^{\frac{1}{2}\int \frac{d(t^2+1)}{t^2+1}} = e^{\frac{1}{2}\ln(t^2+1)} = (t^2+1)^{1/2}.$$

Therefore, multiplying through the DE by the integrating factor yields

$$(*) \qquad (t^2+1)^{1/2} y' + t(t^2+1)^{-1/2} y = t(t^2+1)^{-1/2}.$$

Observe that the left member of this last equation is exact because

$\frac{d}{dt}\left[(t^2+1)^{1/2} y\right] = (t^2+1)^{1/2} y' + \frac{1}{2}(t^2+1)^{-1/2}(2t)y = (t^2+1)^{1/2} y' + t(t^2+1)^{-1/2} y$. Thus $(*)$ is

$$\frac{d}{dt}\left[(t^2+1)^{1/2} y\right] = t(t^2+1)^{-1/2}.$$

Integrating both sides produces

$$(t^2+1)^{1/2} y = \int t(t^2+1)^{-1/2} dt = \frac{1}{2}\int (t^2+1)^{-1/2} d(t^2+1) = \frac{1}{\cancel{2}} \cdot \cancel{2}(t^2+1)^{1/2} + c$$

and thus

$$\boxed{y(t) = 1 + c(t^2+1)^{-1/2}}$$

where c is an arbitrary constant.

3. (a) [6] According to Newton's law of cooling, the temperature u of an object changes with time at a rate proportional to the difference between its temperature and that of its surroundings T_0. Write, **BUT DO NOT SOLVE,** a differential equation that expresses Newton's law of cooling.

Rate of change of u is proportional to $u - T_0$ so

$$\boxed{\frac{du}{dt} = k(u - T_0)}$$ where k is a constant of proportionality.

(b) [15] Suppose that the temperature of a mug of coffee obeys Newton's law of cooling. If the coffee has a temperature of 100 degrees Celsius when freshly poured and $\ln(3/2)$ hours later has cooled to 75 degrees Celsius in a room at 25 degrees Celsius, find the coffee's temperature at all times $t \geq 0$.

We need to solve the DE in (a) subject to the conditions $u(0) = 100$ and $u(\ln(3/2)) = 75$. Separating variables in the DE in (a) gives

$$\ln|u - T_0| = \int \frac{du}{u - T_0} = \int k\,dt = kt + C$$

where C is an arbitrary constant. Exponentiating produces

$$|u - T_0| = e^{kt+C} = Ae^{kt}$$

where $A = e^C$. But $T_0 = 25$ and $u(t) > 25$ in our case so

$$u(t) = 25 + Ae^{kt}.$$

From the first condition we have $100 = u(0) = 25 + Ae^0$ so $A = 75$. The second condition implies $75 = u(\ln(3/2)) = 25 + 75e^{k\ln(3/2)}$ so

$\frac{50}{75} = e^{k\ln(3/2)}$ or equivalently $\ln\left(\frac{2}{3}\right) = k\ln\left(\frac{3}{2}\right)$ so $k = -1$. Thus

$$\boxed{u(t) = 25 + 75e^{-t}}$$

for $t \geq 0$. Here u is in degrees Celsius and t is in hours.

<u>Alternate solution</u> of the DE in (a) using first-order linear techniques.

Observe that the DE in (a) can be rewritten as

(cont.)

$$\frac{du}{dt} - ku = -kT_0 \, .$$

An integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int -k dt} = e^{-kt + \cancel{c}^{\,0}}$.

Multiplying through the DE by the integrating factor yields

$$(*) \qquad e^{-kt}\frac{du}{dt} - ke^{-kt}u = -kT_0 e^{-kt} .$$

Note that $\frac{d}{dt}\left[e^{-kt}u\right] = e^{-kt}\frac{du}{dt} - ke^{-kt}u$ so the left member of $(*)$ is exact. Hence $(*)$ is equivalent to

$$\frac{d}{dt}\left[e^{-kt}u\right] = -kT_0 e^{-kt} .$$

Integrating both sides of this last equation produces

$$e^{-kt}u = \int -kT_0 e^{-kt}dt = T_0 e^{-kt} + A$$

where A is an arbitrary constant. Multiplying through by e^{kt} gives

$$u(t) = T_0 + Ae^{kt} .$$

The remainder of the solution to (b) follows as before.

Normalize: $y'' - \frac{2}{t^2}y = \frac{3t^2-1}{t^2} = 3 - t^{-2}$ $g(t)$

4.[21] Solve $t^2y'' - 2y = 3t^2 - 1$ on the interval $t > 0$ using variation of parameters.

This is a second-order nonhomogeneous Euler equation: $at^2y'' + bty' + cy = f(t)$. The general solution is $y(t) = y_c(t) + y_p(t)$ where y_c is the general solution of the associated homogeneous equation $t^2y'' - 2y = 0$ and y_p is any particular solution of the nonhomogeneous equation. $y = t^m$ in $t^2y'' - 2y = 0$ leads to $m(m-1) - 2 = 0$ or equivalently $m^2 - m - 2 = 0$ which factors: $(m-2)(m+1) = 0$. Consequently $m = 2$ or $m = -1$ so $y_c(t) = c_1t^2 + c_2t^{-1}$ where c_1 and c_2 are arbitrary constants. Note that

$$W(t^2, t^{-1}) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3 \neq 0 \quad \text{if} \quad t > 0.$$

The variation of parameters formula gives a particular solution of the nonhomogeneous equation of the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = t^2u_1(t) + t^{-1}u_2(t)$$

where

$$u_1 = \int \frac{-y_2 g}{W}\,dt = \int \frac{-t^{-1}(3 - t^{-2})}{-3}\,dt = \int\left(t^{-1} - \frac{1}{3}t^{-3}\right)dt = \ln(t) + \frac{1}{6t^2} + \cancel{C}^{\,0}$$

and

$$u_2 = \int \frac{y_2 g}{W}\,dt = \int \frac{t^2(3 - t^{-2})}{-3}\,dt = \int\left(-t^2 + \frac{1}{3}\right)dt = \frac{t}{3} - \frac{t^3}{3} + \cancel{C}^{\,0}.$$

Therefore $y_p(t) = t^2\left(\ln(t) + \frac{1}{6t^2}\right) + \frac{1}{t}\left(\frac{t}{3} - \frac{t^3}{3}\right) = t^2\ln(t) - \frac{t^2}{3} + \frac{1}{2}$.

The general solution on $t > 0$ is

$$\boxed{y(t) = c_1t^2 + c_2t^{-1} + t^2\ln(t) - \frac{t^2}{3} + \frac{1}{2}}.$$

5.[20] Find the general solution of $y''' - y' = 5t$.

This is a third-order nonhomogeneous linear DE with constant coefficients. The general solution is $y(t) = y_c(t) + y_p(t)$ where y_c is the general solution of the associated homogeneous equation $y''' - y' = 0$ and y_p is any particular solution of the nonhomogeneous equation. $y = e^{rt}$ in $y''' - y' = 0$ leads to $r^3 - r = 0$ which factors: $r(r^2-1) = 0$ or $r(r-1)(r+1) = 0$. Therefore $r = 0$, $r = 1$, or $r = -1$. Hence $y_c(t) = c_1 + c_2 e^{t} + c_3 e^{-t}$ where c_1, c_2, and c_3 are arbitrary constants. It is easier to use the method of undetermined coefficients to find a particular solution. Since $g(t) = 5t$, a trial form for y_p is

$$y_p(t) = t^s(At+B)$$

where A and B are constants to be determined and s is the multiplicity of 0 as a root of the characterstic equation $r^3 - r = 0$. From the calculation above $s = 1$. Therefore $y_p(t) = At^2 + Bt$ so $y_p' = 2At + B$, $y_p'' = 2A$, and $y_p''' = 0$. We want $y_p''' - y_p' = 5t$, so substituting yields $0 - (2At+B) = 5t$, and it follows that $A = -\frac{5}{2}$ and $B = 0$. That is, $y_p(t) = -\frac{5}{2}t^2$ so $\boxed{y(t) = c_1 + c_2 e^{t} + c_3 e^{-t} - \frac{5}{2}t^2.}$

Alternate Method for a Particular Solution Using Variation of Parameters.

A particular solution is

$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + u_3(t)y_3(t) = 1 \cdot u_1(t) + e^{t}u_2(t) + e^{-t}u_3(t)$ where

$u_1 = \int \frac{W_1 g}{W}\,dt$, $u_2 = \int \frac{W_2 g}{W}\,dt$, and $u_3 = \int \frac{W_3 g}{W}\,dt$. Here we have

(cont.)

$$W = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 1 \cdot \begin{vmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{vmatrix} = 2$$

$$W_1 = \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{vmatrix} = 1 \cdot \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2$$

$$W_2 = \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & -e^{-t} \\ 1 & e^{-t} \end{vmatrix} = e^{-t}$$

$$W_3 = \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} e^t & 0 \\ e^t & 1 \end{vmatrix} = e^t .$$

Thus $u_1 = \int \frac{-2(5t)}{2} dt = -\frac{5}{2}t^2 + \cancel{c}^{\,0}$,

$$u_2 = \int \frac{e^{-t}(5t)}{2} dt = \frac{5}{2}\int \underbrace{t}_{u} \underbrace{e^{-t} dt}_{dv} = \frac{5}{2}\left(-te^{-t} - \int -e^{-t} dt\right) = \frac{5}{2}(-t-1)e^{-t} + \cancel{c}^{\,0}$$

$$u_3 = \int \frac{e^{t}(5t)}{2} dt = \frac{5}{2}\int \underbrace{t}_{u} \underbrace{e^{t} dt}_{dv} = \frac{5}{2}\left(te^{t} - \int e^{t} dt\right) = \frac{5}{2}(t-1)e^{t} + \cancel{c}^{\,0} .$$

Therefore $y_p(t) = 1\cdot\left(-\frac{5}{2}t^2\right) + \cancel{e^{t}}\left(\frac{5}{2}\right)(-t-1)\cancel{e^{-t}} + \cancel{e^{-t}}\left(\frac{5}{2}\right)(t-1)\cancel{e^{t}}$

$$= -\frac{5}{2}t^2 + \frac{5}{2}\left(-\cancel{t} - 1 + \cancel{t} - 1\right)$$

$$= -\frac{5}{2}t^2 - 5$$

The rest of the solution proceeds as in the undetermined coefficients calculation:

$$y = y_c + y_p .$$

6. (a) [17] Solve the initial value problem $y'' + 2y' + 2y = \cos(t)\delta(t-\pi)$, $y(0) = 0$, $y'(0) = 1$.

(b) [4] Which is greater, $y(\pi/2)$ or $y(3\pi/2)$? Justify your answer.

(a) Because of the Dirac delta in the driver $g(t) = \cos(t)\delta(t-\pi)$, we use the method of Laplace transforms. If $y = y(t)$ is a solution then

$$y''(t) + 2y'(t) + 2y(t) = \cos(t)\delta(t-\pi),$$

so taking the Laplace transform of both sides and using (6) in the table gives

$$(*) \quad s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) + 2\big(s\mathcal{L}\{y\}(s) - y(0)\big) + 2\mathcal{L}\{y\}(s) = \mathcal{L}\{\cos(t)\delta(t-\pi)\}(s).$$

To evaluate the right member of the equation above, we use the definition of the Laplace transform, $\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt$, and the sifting property of the Dirac delta, $\int_{-\infty}^{\infty} h(t)\delta(t-c)\,dt = h(c)$, for all bounded piecewise continuous functions h on $(-\infty,\infty)$. Let $h_s(t) = \begin{cases} \cos(t)\,e^{-st} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$ Then for $s > 0$,

$$\mathcal{L}\{\cos(t)\delta(t-\pi)\}(s) = \int_0^\infty \cos(t)\delta(t-\pi)e^{-st}\,dt = \int_{-\infty}^{\infty} h_s(t)\,\delta(t-\pi)\,dt = h_s(\pi) = \overset{-1}{\cancel{\cos(\pi)}}\,e^{-s\pi}.$$

Substituting this result, together with the initial conditions $y(0) = 0$, $y'(0) = 1$, into $(*)$ yields

$$s^2\mathcal{L}\{y\}(s) - 1 + 2s\mathcal{L}\{y\}(s) + 2\mathcal{L}\{y\}(s) = -e^{-s\pi}.$$

Rearranging gives

$$(s^2 + 2s + 2)\mathcal{L}\{y\}(s) = 1 - e^{-s\pi}$$

or

$$\mathcal{L}\{y\}(s) = \frac{1}{(s+1)^2 + 1} - e^{-s\pi}\cdot\frac{1}{(s+1)^2 + 1}.$$

Taking the inverse Laplace transform of both sides and using

(cont.)

formula 7 with $F(s+1) = \frac{1}{(s+1)^2+1}$ and formula 8 in the Laplace transform table gives

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\frac{1}{(s+1)^2+1}\right\}$$

$$\boxed{y(t) = e^{-t}\sin(t) - u_\pi(t)\,e^{-(t-\pi)}\sin(t-\pi)\,.}$$

(b) $$y\left(\tfrac{\pi}{2}\right) = e^{-\pi/2}\underbrace{\sin(\pi/2)}_{1} - \cancel{u_\pi}^{\,0}(\pi/2)\,e^{\pi/2}\sin(-\pi/2) = e^{-\pi/2} > 0.$$

$$y\left(\tfrac{3\pi}{2}\right) = e^{-3\pi/2}\underbrace{\sin\left(\tfrac{3\pi}{2}\right)}_{-1} - \cancel{u_\pi}^{\,1}\left(\tfrac{3\pi}{2}\right)e^{-\pi/2}\underbrace{\sin(\pi/2)}_{1} = -e^{-3\pi/2} - e^{-\pi/2} < 0.$$

Therefore $\boxed{y\left(\tfrac{\pi}{2}\right) > y\left(\tfrac{3\pi}{2}\right).}$

7. (a) [18] Find the solution of

$$y'' + 9y = \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ -18 & \text{if } \pi \le t, \end{cases}$$

satisfying $y(0) = 2$, $y'(0) = 0$.

(b) [3] Write your solution as a piecewise defined function.

Method I: Laplace transforms.

(a) We rewrite the DE as $y'' + 9y = -18\,u_\pi(t)$. Taking the Laplace transform of both sides gives

$$s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) + 9\mathcal{L}\{y\}(s) = -18\frac{e^{-\pi s}}{s}$$

by formulas 6, 8, and 1 of the Laplace transform table. Applying the initial conditions $y(0)=2$ and $y'(0)=0$ and rearranging gives

$$(*) \qquad \mathcal{L}\{y\}(s) = \frac{2s}{s^2+9} - e^{-\pi s} \cdot \frac{18}{s(s^2+9)}.$$

The partial fraction decomposition proceeds as follows:

$$\frac{18}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9} \implies 18 = A(s^2+9) + (Bs+c)s.$$

Take $s=0$ to find A: $\quad 18 = A(9)$ so $A = 2$.

Take $s = 3i$ to find B and C: $\quad 18 = (3iB + C)(3i) = -9B + 3Ci \Rightarrow B = -2, C = 0.$

Substituting in (*) and taking the inverse Laplace transform yields

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+9}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\left(\frac{2}{s} - \frac{2s}{s^2+9}\right)\right\}$$

$$\boxed{y(t) = 2\cos(3t) - u_\pi(t)\left[2 - 2\cos(3(t-\pi))\right]}$$

by formulas 4, 1, and 8 of the Laplace transform table.

(b) Using the definition of the unit step function $u_\pi(t) = \begin{cases} 0 & \text{if } t < \pi, \\ 1 & \text{if } t \ge \pi, \end{cases}$ we have

$$\boxed{y(t) = \begin{cases} 2\cos(3t) & \text{if } 0 \le t < \pi, \\ \cancel{2\cos(3t)} - 2 + \cancel{2\cos(3(t-\pi))} & \text{if } \pi \le t. \end{cases}}$$

<u>Method II</u>: Solve the IVP on $0 \le t < \pi$ and use continuity to extend the solution to $\pi \le t < \infty$.

(a) If $y'' + 9y = 0$ on $0 \le t < \pi$ then $y = e^{rt}$ leads to $r^2 + 9 = 0$ so $r = \pm 3i$. Hence $y(t) = c_1\cos(3t) + c_2\sin(3t)$ is the general solution of the DE on $0 \le t < \pi$. Applying the initial conditions yields

$$2 = y(0) = c_1\cos(0) + c_2\sin(0) = c_1$$

and

$$0 = y'(0) = -3c_1\sin(3t) + 3c_2\cos(3t)\Big|_{t=0} = -3c_1\sin(0) + 3c_2\cos(0) = 3c_2.$$

Thus $\boxed{y(t) = 2\cos(3t) \text{ if } 0 \le t < \pi.}$ Note that $\lim_{t\to\pi^-} y(t) = 2\cos(3\pi) = -2$ and $\lim_{t\to\pi^-} y'(t) = \lim_{t\to\pi^-} -6\sin(3t) = -6\sin(3\pi) = 0$. Therefore on $\pi \le t < \infty$ we must solve the IVP: $y'' + 9y = -18$, $y(\pi) = -2$, $y'(\pi) = 0$. The general solution of the nonhomogeneous DE is $y = y_c + y_p$ where y_c is the general solution of $y'' + 9y = 0$ – i.e. $y_c(t) = c_1\cos(3t) + c_2\sin(3t)$ by the computation above – and y_p is any particular solution of $y'' + 9y = -18$. Noting that $g(t) = -18$, the method of undetermined coefficients suggests $y_p(t) = A$ where A is a constant to be determined. Then $y_p' = 0 = y_p''$ so $y_p'' + 9y_p = -18$ implies $9A = -18$ and consequently $A = -2$. therefore $y(t) = c_1\cos(3t) + c_2\sin(3t) - 2$ if $\pi \le t < \infty$. Applying the initial conditions,

$$-2 = y(\pi) = c_1\cos(3\pi) + c_2\sin(3\pi) - 2 = -c_1 - 2 \qquad \text{so } c_1 = 0;$$

$$0 = y'(\pi) = -3c_1\sin(3t) + 3c_2\cos(3t)\Big|_{t=\pi} = -3c_1\sin(3\pi) + 3c_2\cos(3\pi) = -3c_2.$$

Thus $\boxed{y(t) = -2 \text{ on } \pi \le t < \infty.}$

(b)

$$\boxed{y(t) = \begin{cases} 2\cos(3t) & \text{if } 0 \le t < \pi, \\ -2 & \text{if } \pi \le t < \infty. \end{cases}}$$

8.[21] Solve the initial value problem $\mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 2 & -5 \end{pmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Method I: Matrix method.

Let $A = \begin{bmatrix} -1 & -2 \\ 2 & -5 \end{bmatrix}$. Then $\vec{x} = \vec{k}e^{rt}$ in $\vec{x}' = A\vec{x}$ leads to $r\vec{k}\cancel{e^{rt}} = A\vec{k}\cancel{e^{rt}}$ so r is an eigenvalue of A and $\vec{k}$ is a corresponding eigenvector of A. Then

$$0 = |A - rI| = \begin{vmatrix} -1-r & -2 \\ 2 & -5-r \end{vmatrix} = (r+5)(r+1) + 4 = r^2 + 6r + 9 = (r+3)^2.$$

An eigenvector $\vec{k}$ of A corresponding to $r = -3$ satisfies $(A - rI)\vec{k} = \vec{0}$

so $\begin{bmatrix} -1-(-3) & -2 \\ 2 & -5-(-3) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or equivalently $\begin{cases} 2k_1 - 2k_2 = 0, \\ \cancel{2k_1 - 2k_2 = 0}. \text{ Redundant} \end{cases}$

Therefore $k_2 = k_1$ so $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Consequently $\vec{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-3t}$ is one solution to $\vec{x}' = A\vec{x}$. To obtain a second linearly independent in this case of the repeated real eigenvalue $r = -3$, we assume $\vec{x}(t) = \vec{k}te^{rt} + \vec{l}e^{rt}$. Substituting in $\vec{x}' = A\vec{x}$ leads to the system

$$\begin{cases} (A - rI)\vec{k} = \vec{0} \\ (A - rI)\vec{l} = \vec{k}. \end{cases}$$

We have already solved the first equation of the system and found $r = -3$, $\vec{k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Substituting into the second equation of the system yields

$$\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Clearly $2l_1 - 2l_2 = 1$ so $l_1 = l_2 + \frac{1}{2}$. For convenience, we take $l_2 = 0$ so $\vec{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$. Therefore the general solution of $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-3t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}te^{-3t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}e^{-3t}\right)$$ where c_1 and c_2 are arbitrary consts.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ implies $c_1 = 0, c_2 = 2$. Thus $\boxed{\vec{x}(t) = 2te^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

solves the initial value problem.

Method II : Substitution Method.

We rewrite the problem as a coupled system of scalar DEs:

$$(*)\quad \begin{cases} x_1' = -x_1 - 2x_2, & x_1(0) = 1, \\ x_2' = 2x_1 - 5x_2, & x_2(0) = 0. \end{cases}$$

Solving the first equation of (*) for x_2 yields

$$(\dagger)\qquad x_2 = -\tfrac{1}{2}(x_1' + x_1).$$

Substituting this expression for x_2 in the second equation of (*), we have

$$-\tfrac{1}{2}(x_1' + x_1)' = 2x_1 + \tfrac{5}{2}(x_1' + x_1).$$

Simplifying and rearranging gives

$$-(x_1'' + x_1') = 4x_1 + 5(x_1' + x_1)$$

or

$$(**)\qquad 0 = x_1'' + 6x_1' + 9x_1.$$

If $x_1(t) = e^{rt}$ in (**) then $0 = r^2 + 6r + 9 = (r+3)^2$ so $r = -3$ is a root with multiplicity two. Hence

$$(\square)\qquad x_1(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

is the general solution of (**) where c_1 and c_2 are arbitrary constants.

Note that the first initial condition in (*) implies

$$1 = x_1(0) = c_1 e^{-3(0)} + c_2(0) e^{-3(0)} = c_1.$$

Employing the first equation of (*) and the initial conditions produces

$$x_1'(0) = -x_1(0) - 2x_2(0) = -1 - 2(0) = -1,$$

and differentiating ($\square$) gives

(cont.)

$$x_1'(t) = -3c_1 e^{-3t} + c_2\left(-3te^{-3t} + e^{-3t}\right).$$

Therefore

$$-1 = x_1'(0) = -3c_1 + c_2 = -3(1) + c_2 \qquad \text{so} \qquad c_2 = 2.$$

Thus (□) yields

$$x_1(t) = e^{-3t} + 2te^{-3t}$$

and

$$x_1'(t) = -3e^{-3t} + 2\left(-3te^{-3t} + e^{-3t}\right) = -e^{-3t} - 6te^{-3t}.$$

Using (†) then gives

$$x_2(t) = -\frac{1}{2}\left(-e^{-3t} - 6te^{-3t} + e^{-3t} + 2te^{-3t}\right) = 2te^{-3t}.$$

That is,

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-3t} + 2te^{-3t} \\ 2te^{-3t} \end{bmatrix}$$

or

$$\boxed{\vec{x}(t) = 2te^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}.$$

<u>Method III</u>: The Laplace transform method.

We rewrite the IVP as a coupled system of scalar DEs:

$$(*) \quad \begin{cases} x_1' = -x_1 - 2x_2, & x_1(0) = 1, \\ x_2' = 2x_1 - 5x_2, & x_2(0) = 0. \end{cases}$$

(cont.)

Taking the Laplace transform of each equation in (*) gives

$$\begin{cases} s\mathcal{L}\{x_1\}(s) - \cancel{x_1(0)}^{\,1} = -\mathcal{L}\{x_1\}(s) - 2\mathcal{L}\{x_2\}(s) \\ s\mathcal{L}\{x_2\}(s) - \cancel{x_2(0)}_{\,0} = 2\mathcal{L}\{x_1\}(s) - 5\mathcal{L}\{x_2\}(s) \end{cases}$$

and rearranging, we have

$$\begin{cases} (s+1)\mathcal{L}\{x_1\}(s) + 2\mathcal{L}\{x_2\}(s) = 1 \\ -2\mathcal{L}\{x_1\}(s) + (s+5)\mathcal{L}\{x_2\}(s) = 0. \end{cases}$$

Solving the second equation of the last system for $\mathcal{L}\{x_1\}(s)$ gives

$$\mathcal{L}\{x_1\}(s) = \frac{s+5}{2}\mathcal{L}\{x_2\}(s)$$

and substituting into the first equation yields

$$(s+1)\frac{(s+5)}{2}\mathcal{L}\{x_2\}(s) + 2\mathcal{L}\{x_2\}(s) = 1.$$

Simplifying and rearranging produces

$$(s+1)(s+5)\mathcal{L}\{x_2\}(s) + 4\mathcal{L}\{x_2\}(s) = 2$$

$$(s^2+6s+9)\mathcal{L}\{x_2\}(s) = 2$$

$$\mathcal{L}\{x_2\}(s) = \frac{2}{s^2+6s+9} = \frac{2}{(s+3)^2}.$$

Taking the inverse Laplace transform and using formulas 7 and 2 in the Laplace transform table, we have

$$\boxed{x_2(t) =} \mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2}\right\} = 2e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = \boxed{2te^{-3t}}$$

Then $x_2'(t) = -6te^{-3t} + 2e^{-3t}$ so substituting into the second equation of (*) leads to

(cont.)

$$-6te^{-3t} + 2e^{-3t} = 2x_1(t) - 5\left(2te^{-3t}\right).$$

Simplifying and rearranging gives

$$\boxed{2te^{-3t} + e^{-3t} = x_1(t).}$$

As in Method II this is equivalent to

$$\boxed{\vec{x}(t) =} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2te^{-3t} + e^{-3t} \\ 2te^{-3t} \end{bmatrix} = \boxed{2te^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.}$$

9.[21] Find the general solution of the system $x' = 2x + 3y - 7$, $y' = -x - 2y + 5$. Method I: Matrix method

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, and $\vec{g}(t) = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$. Then the system can be rewritten as $\vec{x}\,' = A\vec{x} + \vec{g}(t)$. It is clearly linear and nonhomogeneous with general solution $\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t)$ where $\vec{x}_c$ is the general solution of the associated homogeneous system $\vec{x}\,' = A\vec{x}$ and $\vec{x}_p$ is any particular solution of the nonhomogeneous system. $\vec{x} = \vec{k}e^{rt}$ in $\vec{x}\,' = A\vec{x}$ leads to $r\vec{k} = A\vec{k}$ so we seek eigenvalues r and eigenvectors $\vec{k}$ of A.

$$0 = \det(A - rI) = \begin{vmatrix} 2-r & 3 \\ -1 & -2-r \end{vmatrix} = (r+2)(r-2) + 3 = r^2 - 1 = (r-1)(r+1) \text{ so } r = \pm 1.$$

An eigenvector $\vec{k}$ of A corresponding to $r = 1$ satisfies $(A - 1\cdot I)\vec{k} = \vec{0}$, i.e.

$$\begin{bmatrix} 2-1 & 3 \\ -1 & -2-1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or equivalently } \begin{cases} k_1 + 3k_2 = 0, \\ \cancel{-k_1 - 3k_2 = 0}. \text{ redundant} \end{cases} \quad \text{Thus } k_1 = -3k_2 \text{ so}$$

$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -3k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} -3 \\ 1 \end{bmatrix}. \text{ Consequently } \vec{x}^{(1)}(t) = \vec{k}^{(1)}e^{r_1 t} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{t} \text{ solves } \vec{x}\,' = A\vec{x}.$$

An eigenvector $\vec{k}$ of A corresponding to $r = -1$ satisfies $(A - (-1)\cdot I)\vec{k} = 0$, or

$$\begin{bmatrix} 2-(-1) & 3 \\ -1 & -2-(-1) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ which is equivalent to } \begin{cases} \cancel{3k_1 + 3k_2 = 0} \text{ redundant} \\ -k_1 - k_2 = 0 \quad \text{so } k_2 = -k_1. \end{cases}$$

Hence $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus $\vec{x}^{(2)}(t) = \vec{k}^{(2)}e^{r_2 t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$ solves $\vec{x}\,' = A\vec{x}$.

Since $W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} -3e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{vmatrix} = 2 \neq 0$, $\vec{x}_c(t) = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$.

Because $\vec{g}(t) = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$ is a constant vector, the method of undetermined coefficients with $\vec{x}_p(t) = \vec{k}$, a constant vector, is the easier method for finding a particular solution to $\vec{x}\,' = A\vec{x} + \vec{g}(t)$. Then $\vec{x}_p\,' = \vec{0}$ so substituting gives

$$\vec{0} = A\vec{x}_p + \begin{bmatrix} -7 \\ 5 \end{bmatrix} = A\vec{k} + \begin{bmatrix} -7 \\ 5 \end{bmatrix} \text{ so } A\vec{k} = \begin{bmatrix} 7 \\ -5 \end{bmatrix} \text{ and } \vec{k} = A^{-1}\begin{bmatrix} 7 \\ -5 \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \end{bmatrix}.$$

(cont.)

$\therefore \ \vec{x}_p = \vec{k} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Therefore $\vec{x} = \vec{x}_c(t) + \vec{x}_p(t) =$

$c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is the general solution of $\vec{x}' = A\vec{x} + \vec{g}(t)$.

That is,

$$\boxed{\begin{aligned} x(t) &= -3c_1e^{t} + c_2e^{-t} - 1 \\ y(t) &= \quad c_1e^{t} - c_2e^{-t} + 3 \end{aligned}}$$

solves the system; here c_1 and c_2 are arbitrary constants.

<u>Alternate Method</u> for a Particular Solution Using Variation of Parameters.

A particular solution is given by $\vec{x}_p(t) = \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\vec{g}(s)\,ds$ where $\Psi(t)$ is a fundamental matrix for $\vec{x}' = A\vec{x}$. In our case, the work above shows that $\Psi(t) = \begin{bmatrix} \vec{x}^{(1)}(t) & \vec{x}^{(2)}(t) \end{bmatrix} = \begin{bmatrix} -3e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}$. Hence

$$\Psi^{-1}(t) = \frac{1}{\det \Psi(t)}\begin{bmatrix} \psi_{22}(t) & -\psi_{12}(t) \\ -\psi_{21}(t) & \psi_{11}(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^{t} & -3e^{t} \end{bmatrix}, \text{ so}$$

$$\int_{t_0}^{t} \Psi^{-1}(s)\vec{g}(s)\,ds = \int_{t_0}^{t} -\frac{1}{2}\begin{bmatrix} e^{-s} & e^{-s} \\ e^{s} & 3e^{s} \end{bmatrix}\begin{bmatrix} -7 \\ 5 \end{bmatrix}ds = \int_{t_0}^{t}\begin{bmatrix} e^{-s} \\ -4e^{s} \end{bmatrix}ds = \begin{bmatrix} -e^{-t} \\ -4e^{t} \end{bmatrix} + \overset{\vec{0}}{\cancel{\vec{c}}}.$$

Therefore $\vec{x}_p(t) = \Psi(t)\begin{bmatrix} -e^{-t} \\ -4e^{t} \end{bmatrix} = \begin{bmatrix} -3e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} -e^{-t} \\ -4e^{t} \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

The solution is finished in the same way as above.

Method II: Substitution Method. To solve the system

$$(*) \qquad \begin{cases} x' = 2x + 3y - 7, \\ y' = -x - 2y + 5, \end{cases}$$

solve the second equation of (*) for x:

$$(\dagger) \qquad x = -y' - 2y + 5$$

and substitute into the first equation of (*):

$$(-y' - 2y + 5)' = 2(-y' - 2y + 5) + 3y - 7.$$

Simplifying and rearranging yields

$$-y'' - 2y' = -2y' - 4y + 10 + 3y - 7$$

or

$$(\dagger\dagger) \qquad y'' - y = -3.$$

The general solution is $y = y_c(t) + y_p(t)$ where y_c is the general solution of the associated homogeneous equation $y'' - y = 0$ and y_p is any particular solution of the nonhomogeneous equation. Then $y(t) = e^{rt}$ in $y'' - y = 0$ leads to $0 = r^2 - 1 = (r-1)(r+1)$ so $y_c(t) = c_1 e^{t} + c_2 e^{-t}$. Employing the method of undetermined coefficients to find a particular solution, a candidate for a particular solution is $y_p(t) = A$ where A is a constant to be determined. Then $y_p' = 0 = y_p''$ so substituting into ($\dagger\dagger$) we have

$$-3 = y_p'' - y_p = 0 - A \qquad \text{so} \qquad A = 3.$$

Hence $\boxed{y(t) =} \, y_c(t) + y_p(t) = \boxed{c_1 e^{t} + c_2 e^{-t} + 3}$. Substituting in ($\dagger$) yields

$$x(t) = -y'(t) - 2y(t) + 5 = -(c_1 e^{t} - c_2 e^{-t}) - 2(c_1 e^{t} + c_2 e^{-t} + 3) + 5$$

or $\boxed{x(t) = -3c_1 e^{t} - c_2 e^{-t} - 1}$ where c_1, c_2 are arbitrary consts.

10. [20] Solve the initial value problem $ty'' + y' = 1 + \frac{1}{(t-2)^2}$, $y(1) = 0$, $y'(1) = 3$.

Method I: Reduction of order.

(1) Let $u = y'$. Then $u' = y''$ so the DE is equivalent to the first-order linear DE $tu' + u = 1 + (t-2)^{-2}$. Normalizing, we have

(1) $u' + \frac{1}{t}u = \frac{1}{t} + \frac{1}{t(t-2)^2}$, so an integrating factor is $\mu = e^{\int p(t)dt} =$

(2) $e^{\int \frac{1}{t}dt} = e^{\ln(t) + \cancel{c}^{0}} = t$. Multiplying the normalized DE by the integrating factor produces a DE whose left member is exact:

(1) $$\left(\frac{d}{dt}\left[t u \right] = \right) tu' + u = 1 + \frac{1}{(t-2)^2} .$$

Integrating once gives

(3) $$tu = \int \left(1 + \frac{1}{(t-2)^2} \right) dt = t - \frac{1}{t-2} + c_1$$

or

$$(y' =) \quad u = 1 - \frac{1}{t(t-2)} + \frac{c_1}{t}$$ (8pts. to here)

Integrating again produces

(1) $$y(t) = \int \left(1 - \frac{1}{t(t-2)} + \frac{c_1}{t} \right) dt .$$

A partial fraction decomposition computation proceeds as follows.

(2) $$\frac{1}{t(t-2)} = \frac{A}{t} + \frac{B}{t-2} \implies 1 = A(t-2) + Bt.$$

To find A, set $t=0$: $1 = A(-2)$ so $A = -1/2$.

To find B, set $t=2$: $1 = B(2)$ so $B = 1/2$.

(cont.)

(11 pts. to here)

(11 pts. to here)

(1) Therefore $y(t) = \int \left[1 - \left(\frac{-\frac{1}{2}}{t} + \frac{\frac{1}{2}}{t-2}\right) + \frac{c_1}{t}\right] dt$

(5) or $y(t) = t + \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t-2| + c_1 \ln|t| + c_2$.

Applying the initial conditions $y(1) = 0$ and $y'(1) = 3$ yields

(1) $0 = y(1) = 1 + c_2$ so $c_2 = -1$;

(1) $3 = y'(1) = 1 - \frac{1}{1(1-2)} + \frac{c_1}{1}$ so $c_1 = 1$.

(1) Therefore $\boxed{y(t) = t + \frac{3}{2}\ln(t) - \frac{1}{2}\ln(2-t) - 1}$ for $0 < t < 2$.

<u>Method II</u>: Transform into an Euler equation.

Multiplying through $ty'' + y' = 1 + \frac{1}{(t-2)^2}$ by t yields a nonhomogeneous Euler equation:

(1) $$t^2 y'' + ty' = t + \frac{t}{(t-2)^2}.$$

The general solution is $y(t) = y_c(t) + y_p(t)$ where y_c is the general solution of the associated homogeneous equation $t^2y'' + ty' = 0$ and y_p is any particular solution of the nonhomogeneous equation. Now

(1) $y = t^m$ in $t^2y'' + ty' = 0$ leads to $m(m-1) + m = 0$ or equivalently

(1) $m^2 = 0$, so $m = 0$ (multiplicity two). Consequently

(2) $$y_c(t) = c_1 + c_2 \ln|t|.$$

(5 pts. to here)

(5 pts. to here)

(1) Note that $W(1, \ln|t|) = \begin{vmatrix} 1 & \ln|t| \\ 0 & \frac{1}{t} \end{vmatrix} = \frac{1}{t} \neq 0$ on $t > 0$.

We use variation of parameters to find a particular solution:

(1) $$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = 1 \cdot u_1(t) + (\ln|t|) \cdot u_2(t)$$

where

(1) $$u_1 = \int \frac{-y_2 g}{W}\,dt = \int \frac{-\ln|t|\left(\frac{1}{t} + \frac{1}{t(t-2)^2}\right)}{t^{-1}}\,dt = -\int \overbrace{\ln|t|}^{U}\overbrace{\left(1 + \frac{1}{(t-2)^2}\right)dt}^{dV}$$

$$= -\left(\left(t - \frac{1}{t-2}\right)\ln|t| - \int \left(t - \frac{1}{t-2}\right)\frac{1}{t}\,dt\right)$$

$$= -t\ln|t| + \frac{\ln|t|}{t-2} + \int \left(1 - \frac{1}{t(t-2)}\right)dt$$

$$= -t\ln|t| + \frac{\ln|t|}{t-2} + \int \left(1 + \frac{1/2}{t} - \frac{1/2}{t-2}\right)dt$$

(4) $$= -t\ln|t| + \frac{\ln|t|}{t-2} + t + \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t-2| + \cancelto{0}{c}$$

and

(1) $$u_2 = \int \frac{y_1 g}{W}\,dt = \int \frac{1 \cdot \left(\frac{1}{t} + \frac{1}{t(t-2)^2}\right)}{t^{-1}}\,dt = \int \left(1 + \frac{1}{(t-2)^2}\right)dt$$

(1) $$= t - \frac{1}{t-2} + \cancelto{0}{c}.$$

Consequently

$$y_p(t) = -\cancel{t\ln|t|} + \cancel{\frac{\ln|t|}{t-2}} + t + \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t-2| + \ln|t|\left(\cancel{t} - \cancel{\frac{1}{t-2}}\right)$$

(1) $$= t + \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t-2|.$$

The rest of the solution proceeds as in Method I.

(5 remaining pts.: 2 pts. for $y = y_c + y_p$ and 2 pts. for constants and 1 pt. for writing solution)

A SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\dfrac{1}{s-a}$
2. t^n	$\dfrac{n!}{s^{n+1}}, \quad n = 0,1,2,3....$
3. $\sin(bt)$	$\dfrac{b}{s^2+b^2}$
4. $\cos(bt)$	$\dfrac{s}{s^2+b^2}$
5. $(f*g)(t)$	$F(s)G(s)$
6. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
7. $e^{ct} f(t)$	$F(s-c)$
8. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
9. $\delta(t-c)$	e^{-cs}

Math 204 Final Exam "Master List" Scorecard, 2012 Fall Semester

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201 Fs (58.1%)

Number taking final: 346
Median: 108.5
Mean: 113.2
Standard Deviation: 41.8

Math 204 Final Exam, 2012 Fall Semester, Instructor Grow, **Section** M

200			149			98			47	
199			148			97			46	
198			147	I		96			45	I
197			146			95			44	
196			145			94			43	
195			144		8 Cs	93		11 Fs	42	
194		1 A	143			92			41	
193			142			91		(40.7%)	40	
192		(3.7%)	141	I	(29.6%)	90			39	
191			140			89	I		38	
190			139			88			37	
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187			136			85	I		34	
186			135			84			33	
185	I		134			83	I		32	
184			133			82			31	
183			132			81			30	
182			131			80			29	
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179			128			77			26	
178			127	I		76			25	
177			126			75			24	
176			125			74			23	
175			124			73			22	
174		5 Bs	123		2 Ds	72			21	
173			122			71			20	
172		(18.5%)	121	I	(7.4%)	70			19	
171	I		120			69			18	
170			119			68			17	
169			118			67			16	I
168			117			66			15	
167	II		116			65			14	
166	I		115			64			13	
165			114			63			12	
164			113			62			11	
163			112			61			10	
162			111			60			9	
161	I		110			59			8	
160			109			58			7	
159	I		108			57			6	
158			107	I		56			5	
157			106			55			4	
156	I		105			54			3	
155	I		104			53			2	
154	I		103			52			1	
153			102	I		51			0	
152	I		101	I		50				
151			100	I		49				
150	I		99			48				

Number taking final: 27
Median: 141
Mean: 125.1
Standard Deviation: 42.2