Mathematics 204

Spring 2010

Final Exam

[1] Your Printed Name: Dr. Grow

[1] Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. The final exam consists of this cover page, 8 pages of problems containing 11 numbered problems, and a short table of Laplace transform formulas.
4. Once the exam begins, you will have 120 minutes to complete your solutions.
5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals, partial fraction decompositions, and matrix computations must be done by hand.
6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
7. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 200.

problem	0	1	2	3	4	5	6	7	8	9	10	11	Sum
points earned													
maximum points	2	12	20	20	10	18	21	20	15	20	21*	21	200

* The 1 bonus point is not included in this point value.

1.[12] State the order of each of the following differential equations. Is the equation linear or nonlinear? If the equation is linear, is it homogeneous?

	Order?	Linear?	Homogeneous?
$y''' + ty' + \cos^3(t)y - t^3 = 0$	3	Yes	No
$x'' + t\ln(x)x = 0$	2	No	
$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + y = 0$	3	No	
$y'' + 10y - \delta(t-2) = 0$	2	Yes	No
$\frac{dp}{dt} + p(p-1) = 0$	1	No	

(1^{st} Order, Nonlinear, Separable)

2.[20] Find an explicit solution of the initial value problem $y' = \frac{x}{y}$, $y(0) = -1$.

$$y\,dy = x\,dx$$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \tilde{c}$$

$$y^2 = x^2 + c$$

$$y = \pm\sqrt{x^2+c}$$

In order that y satisfy the initial condition $y(0) = -1$, we must select the minus sign. I.e. $y(x) = -\sqrt{x^2+c}$. Then $-1 = y(0) = -\sqrt{0+c} \Rightarrow c = 1$.

Therefore $\boxed{y(x) = -\sqrt{x^2+1}}$.

3.[20] Solve the differential equation $x^2 y' = 1 - xy$ when $x > 0$.

$$x^2 y' + xy = 1 \qquad (1^{st}\text{-order, Linear, Nonhomogeneous})$$

$$y' + \frac{1}{x} y = \frac{1}{x^2}$$

$$\therefore \quad \mu = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln(x) + \cancel{c}^{\,0}} = x \quad \text{is an integrating factor.}$$

$$\therefore \quad x\left(y' + \frac{1}{x}y\right) = x\left(\frac{1}{x^2}\right)$$

$$\underbrace{xy' + y}_{\text{Exact!}} = \frac{1}{x}$$

$$\frac{d}{dx}\left\{xy\right\} = \frac{1}{x}$$

$$\int \frac{d}{dx}\left\{xy\right\} dx = \int \frac{1}{x} dx$$

$$xy = \ln(x) + c$$

$$\boxed{y = \frac{\ln(x) + c}{x}} \qquad (c \text{ an arbitrary constant})$$

4.[10] According to Newton's empirical law of cooling/warming, the rate of change of the temperature $T(t)$ of a body at time t is proportional to the difference between the temperature of the body and the temperature T_m of the surrounding medium. State this law in the form of a differential equation. Please **DO NOT SOLVE** your differential equation.

$$\boxed{\frac{dT}{dt} = k(T - T_m)}$$

where k is a (negative) constant.

5.[18] A tank initially contains 120 liters of pure water. A solution containing 1.5 grams of salt per liter enters the tank at a rate of 2 liters per minute, and the well-stirred mixture leaves the tank at the same rate. Set up, **BUT DO NOT SOLVE**, an initial value problem that models the amount of salt in the tank at any time $t \geq 0$.

$A(t)$ = the number of grams of salt in the tank at time t

Net Rate = Rate In − Rate Out

$$\frac{dA}{dt} = \left(1.5 \frac{g}{\cancel{L}}\right)\left(2 \frac{\cancel{L}}{min}\right) - \left(\frac{A(t)\, g}{120\, \cancel{L}}\right)\left(2 \frac{\cancel{L}}{min}\right)$$

(All terms have units of g/min.)

$$\boxed{\frac{dA}{dt} = 3 - \frac{1}{60}A, \quad A(0) = 0}$$

6.[21] Find the general solution of $y^{(4)} - 16y = e^x$. You may find the following identity useful:

$$a^4 - b^4 = (a-b)(a+b)(a^2+b^2).$$

$y = e^{mx}$ in $y^{(4)} - 16y = 0$ leads to $m^4 - 16 = 0$

$$(m-2)(m+2)(m^2+4) = 0$$

$$\Rightarrow m = 2, -2, \pm 2i.$$

$$\therefore y_c(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos(2x) + c_4 \sin(2x).$$

$(D^4 - 16)y = e^x$. $(D-1)$ annihilates e^x.

$$\therefore (D-1)(D^4-16)y = (D-1)e^x = 0$$

$$\Rightarrow (D-1)(D-2)(D+2)(D^2+4)y = 0. \quad (\dagger)$$

$y = e^{mx}$ leads to $(m-2)(m-1)(m+2)(m^2+4) = 0$ so $m = 1, 2, -2, \pm 2i$.

The solution to $(\dagger)$ is $y = \underbrace{c_1 e^x}_{y_p?} + \underbrace{c_2 e^{2x} + c_3 e^{-2x} + c_4 \cos(2x) + c_5 \sin(2x)}_{y_c}$.

Therefore $y_p = Ae^x$ where A is a constant to be determined so that

$$y_p^{(4)} - 16 y_p = e^x.$$

$$\Rightarrow Ae^x - 16Ae^x = e^x$$

$$\Rightarrow A = -\frac{1}{15}$$

so $y_p = -\frac{1}{15}e^x$.

The general solution is $y = y_c + y_p$

or $\boxed{y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos(2x) + c_4 \sin(2x) - \frac{1}{15}e^x}$,

where c_1, c_2, c_3, and c_4 are arbitrary constants.

7.[20] Solve the initial value problem $y'' + y = \delta(t-\pi)$, $y(0) = 0$, $y'(0) = 1$.

$$\mathcal{L}\{y'' + y\}(s) = \mathcal{L}\{\delta(t-\pi)\}(s)$$

$$s^2\mathcal{L}\{y''\}(s) - s\overset{0}{\cancel{y(0)}} - \overset{1}{\cancel{y'(0)}} + \mathcal{L}\{y\}(s) = e^{-\pi s}$$

$$(s^2+1)\mathcal{L}\{y\}(s) = 1 + e^{-\pi s}$$

$$\mathcal{L}\{y\}(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}\right\}$$ (Apply formulas 4 and 11 in the short table of Laplace transforms.)

$$y(t) = \sin(t) + \sin(t-\pi)\mathcal{U}(t-\pi)$$

$$y(t) = \begin{cases} \sin(t) & \text{if } 0 \le t < \pi, \\ \sin(t) + \sin(t-\pi) & \text{if } \pi \le t < \infty, \end{cases}$$

$$\boxed{y(t) = \begin{cases} \sin(t) & \text{if } 0 \le t < \pi, \\ 0 & \text{if } \pi \le t < \infty. \end{cases}}$$

$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$\therefore \sin(t-\pi) = \sin(t)\overset{-1}{\cancel{\cos(\pi)}} - \cos(t)\overset{0}{\cancel{\sin(\pi)}} = -\sin(t)$$

8.[15] A spring hangs vertically from a rigid support. When a 4 pound object is attached to the end of the spring, the object stretches the spring $1/2$ foot. Suppose the object is released from rest from a point 1 foot above the equilibrium position. Assuming air resistance (or the damping force) at any instant is equal to one-half the instantaneous velocity of the object, write, **BUT DO NOT SOLVE**, an initial value problem that models the motion of the object if it is driven by an external downward force at time t of $\sin(2t)$ pounds.

$$my'' + \beta y' + ky = f(t)$$

$$\boxed{\begin{gathered} \tfrac{1}{8}y'' + \tfrac{1}{2}y' + 8y = \sin(2t) \\ y(0) = -1, \; y'(0) = 0 \end{gathered}}$$

$mg = \text{weight} \Rightarrow m(32) = 4$

so $m = \frac{1}{8}$ slug

$ks_0 = \text{weight} \Rightarrow k(\frac{1}{2}) = 4$

so $k = 8$

$\beta = \frac{1}{2}$ (from description)

$f(t) = \sin(2t)$

9.[20] Use the Laplace transform method to solve the initial value problem

$$y'' - 4y' + 4y = t^3 e^{2t}, \; y(0) = y'(0) = 0.$$

NO CREDIT will be awarded for any other method of solution.

$$\mathcal{L}\{y'' - 4y' + 4y\}(s) = \mathcal{L}\{t^3 e^{2t}\}(s)$$ (Apply formulas 7, 6, 9, and 3.)

$$s^2\mathcal{L}\{y\}(s) - \cancel{s y(0)}^{0} - \cancel{y'(0)}^{0} - 4\left(s\mathcal{L}\{y\}(s) - \cancel{y(0)}^{0}\right) + 4\mathcal{L}\{y\}(s) = \frac{6}{(s-2)^4}$$

12 pts. to here
(4 + 3 + 1 + 4)

$$(s^2 - 4s + 4)\mathcal{L}\{y\}(s) = \frac{6}{(s-2)^4}$$

13 pts. to here

$$(s-2)^2\mathcal{L}\{y\}(s) = \frac{6}{(s-2)^4}$$

14 pts. to here

$$\mathcal{L}\{y\}(s) = \frac{6}{(s-2)^6}$$

15 pts. to here

$$y(t) = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\} = 6 \cdot e^{2t} \cdot \frac{t^5}{5!}$$

20 pts. to here.

(Apply formulas 9 and 3.)

$$\boxed{y(t) = \frac{t^5 e^{2t}}{20}}$$

10.[21] Solve the initial value problem $\mathbf{X}' = \overbrace{\begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}}^{A}\mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Bonus [1]: Sketch this trajectory of the system in the phase plane.

$\vec{x} = \vec{k}e^{\lambda t}$ leads to $\lambda\vec{k} = A\vec{k}$. $0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & -1/2 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + 1$. 2 pts. to here

4 pts. to here Eigenvalues: $\lambda = \pm i$. Eigenvector corresponding to $\lambda = i$: $(A - \lambda I)\vec{k} = \vec{0}$,

5 pts. to here $\begin{bmatrix} -i & -1/2 \\ 2 & -i \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{cases} \cancel{-ik_1 - \frac{1}{2}k_2 = 0} \\ 2k_1 - ik_2 = 0 \end{cases}$ Redundant: $2i$ times first equation is the second eqn $\Rightarrow k_2 = \frac{2}{i}k_1 = -2ik_1$.

7 pts. to here $\therefore \vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -2ik_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ -2i \end{bmatrix}$ so $\vec{x}^{(1)} = \vec{k}^{(1)}e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}e^{it} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right)(\cos(t) + i\sin(t))$.

9 pts. to here

12 pts. to here $\tilde{\vec{x}}^{(1)} = \mathrm{Re}(\vec{x}^{(1)}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin(t) = \begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}$

15 pts. to here $\tilde{\vec{x}}^{(2)} = \mathrm{Im}(\vec{x}^{(1)}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(t) = \begin{bmatrix} \sin(t) \\ -2\cos(t) \end{bmatrix}$

Therefore this is a fundamental set of solutions to $\vec{x}' = \begin{bmatrix} 0 & -1/2 \\ 2 & 0 \end{bmatrix}\vec{x}$.

$W(\tilde{\vec{x}}^{(1)}, \tilde{\vec{x}}^{(2)}) = \begin{vmatrix} \cos(t) & \sin(t) \\ 2\sin(t) & -2\cos(t) \end{vmatrix} = -2\cos^2(t) - 2\sin^2(t) = -2 \neq 0$.

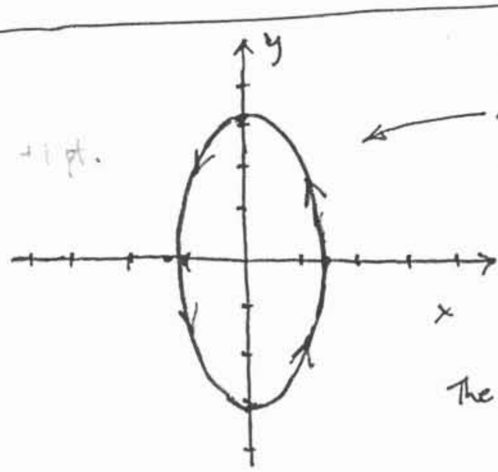
$\therefore \vec{x}(t) = c_1\begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix} + c_2\begin{bmatrix} \sin(t) \\ -2\cos(t) \end{bmatrix}$ is the general solution. 17 pts. to here.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -2 \end{bmatrix} \Rightarrow c_1 = 1, \; c_2 = -1$. 19 pts. to here.

21 pts. to here.

$$\boxed{\therefore \vec{x}(t) = \begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix} - \begin{bmatrix} \sin(t) \\ -2\cos(t) \end{bmatrix} = \begin{bmatrix} \cos(t) - \sin(t) \\ 2\sin(t) + 2\cos(t) \end{bmatrix}}$$ solves the IVP.

+1 pt.



$\begin{cases} x(t) = \cos(t) - \sin(t) \\ y(t) = 2(\sin(t) + \cos(t)) \end{cases}$ $(0 \le t \le 2\pi)$

The trajectory is an ellipse: $x^2 + \frac{y^2}{4} = 2$.

11.[21] Use the method of variation of parameters for systems to solve the system

$$(*)\quad \begin{cases} \dfrac{dx}{dt} = -y + \sec(t) \\ \dfrac{dy}{dt} = x \end{cases}$$

given that the general solution to the associated homogeneous system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}.$$

In vector-matrix form, $(*)$ is $\vec{x}\,' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} \sec(t) \\ 0 \end{bmatrix}$.

The given information is equivalent to saying that $\Phi(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$ is a fundamental matrix for the associated homogeneous system $\vec{x}\,' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$. By variation of parameters, a particular solution to $(*)$ is

$$(**)\qquad \vec{x}_p(t) = \Phi(t) \int^{t} \Phi^{-1}(s) \vec{f}(s)\, ds.$$

$$\Phi^{-1}(s) = \frac{1}{\det \Phi(s)} \begin{bmatrix} \varphi_{22} & -\varphi_{12} \\ -\varphi_{21} & \varphi_{11} \end{bmatrix} = \frac{1}{1} \begin{bmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{bmatrix}.$$

$$\int^{t} \Phi^{-1}(s) \vec{f}(s)\, ds = \int^{t} \begin{bmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{bmatrix} \begin{bmatrix} \sec(s) \\ 0 \end{bmatrix} ds = \int^{t} \begin{bmatrix} 1 \\ -\tan(s) \end{bmatrix} ds = \begin{bmatrix} t + \cancel{c_1}^{\,0} \\ \ln|\cos(t)| + \cancel{c_2}^{\,0} \end{bmatrix}$$

$$\therefore\ \vec{x}_p(t) \overset{(**)}{=} \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} t \\ \ln|\cos(t)| \end{bmatrix} = t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + \ln|\cos(t)| \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

The general solution of $(*)$ is

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\boxed{\vec{x}(t) = (t + c_1) \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + \left(\ln|\cos(t)| + c_2\right) \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}},$$

where c_1 and c_2 are arbitrary constants.

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. 1	$\frac{1}{s}$
2. e^{at}	$\frac{1}{s-a}$
3. t^n	$\frac{n!}{s^{n+1}}, \quad n = 1, 2, 3....$
4. $\sin(kt)$	$\frac{k}{s^2+k^2}$
5. $\cos(kt)$	$\frac{s}{s^2+k^2}$
6. $f'(t)$	$sF(s) - f(0)$
7. $f''(t)$	$s^2F(s) - sf(0) - f'(0)$
8. $\delta(t-t_0)$	e^{-st_0}
9. $e^{at}f(t)$	$F(s-a)$
10. $\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
11. $f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
12. $\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$

no. taking exam: 338

median score: 165

mean score: 158.2

Standard deviation: 32.1