Sec. 7.9 Nonhomogeneous Linear Systems

HW p. 439: # 1, 5, 12, 13 Due: Friday, Dec. 7

Schaum's: ??

In this section we learn how to solve the nonhomogeneous system

$$(*) \qquad \vec{x}' = A\vec{x} + \vec{g}(t).$$

<u>Step 1.</u> Find the general solution $\vec{x}_c(t)$ of the associated homogeneous system

$$(**) \qquad \vec{x}' = A\vec{x}.$$

(We learned how to compute $\vec{x}_c$ in Secs. 7.5 – 7.8:

$$\vec{x}_c(t) = \underbrace{c_1\vec{x}^{(1)}(t) + \dots + c_n\vec{x}^{(n)}(t)}_{\vec{x}^{(1)}, \dots, \vec{x}^{(n)} \text{ is a F.S.S. for } (**)} = \Psi(t)\vec{c} \quad)$$

fundamental matrix of (**)

<u>Step 2:</u> Find a particular solution $\vec{x}_p(t)$ of the nonhomogeneous system (*).

Read text, pp. 432-4 → (a) Diagonalize the coefficient matrix A and solve the "uncoupled" system.
or
Lecture (time permitting) → (b) Use method of undetermined coefficients to find $\vec{x}_p$
or
Lecture → (c) Use variation of parameters to find $\vec{x}_p$
or
Read text, pp. 438-9 → (d) Use Laplace transforms to solve system.

<u>Step 3:</u> Write the general solution: $\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t)$.

(Variation of Parameters when A is a 2x2)

Recall that a particular solution of $y'' + p(t)y' + q(t)y = g(t)$ has the form $y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$ where $\{y_1, y_2\}$ is a F.S.S. to the associated homogeneous equation $y'' + p(t)y' + q(t)y = 0$. By analogy, we expect a particular solution of the nonhomogeneous system (*) to be of the form

$$(+) \qquad \vec{x}_p = u_1(t)\vec{x}^{(1)}(t) + u_2(t)\vec{x}^{(2)}(t)$$

where $\{\vec{x}^{(1)}, \vec{x}^{(2)}\}$ is a F.S.S. of the associated homogeneous system $\vec{x}' = A\vec{x}$.

(†) can be rewritten

$$(\dagger\dagger) \qquad \vec{x}_p = \begin{bmatrix} \vec{x}^{(1)}(t) & \vec{x}^{(2)}(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \Psi(t)\vec{u}(t)$$

where $\Psi(t)$ is a fundamental matrix for $\vec{x}' = A\vec{x}$.

Differentiating (††) gives

$$\vec{x}_p{}' = \Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t).$$

Substituting in (*) $\vec{x}' = A\vec{x} + \vec{g}(t)$ gives

$$\Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t) = \vec{x}_p{}' = A\vec{x}_p + \vec{g}(t) = A\Psi(t)\vec{u}(t) + \vec{g}(t),$$

Rearranging and using the fact that Ψ solves $\vec{x}' = A\vec{x}$ (p. 415) yields

$$\underbrace{\left(\Psi'(t) - A\Psi(t)\right)}_{0}\vec{u}(t) + \Psi(t)\vec{u}'(t) = \vec{g}(t)$$

$$\Psi(t)\vec{u}'(t) = \vec{g}(t)$$

$$\vec{u}'(t) = \Psi(t)^{-1}\vec{g}(t)$$

$$\vec{u}(t) = \int_{t_0}^{t} \Psi^{-1}(s)\vec{g}(s)\,ds$$

$$\therefore \quad \vec{x}_p(t) = \Psi(t)\vec{u}(t)$$

$$\boxed{\vec{x}_p(t) = \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\vec{g}(s)\,ds}$$

Variation of Parameters formula for a particular solution of $\vec{x}' = A\vec{x} + \vec{g}(t)$.
(Here $\Psi(t)$ is a fundamental matrix for $\vec{x}' = A\vec{x}$.)

Ex 1 (Fall 2007 Final Exam)

Given that $\Psi(t) = \begin{bmatrix} t+1 & 1 \\ t & 1 \end{bmatrix}$ is a fundamental matrix on $(0,\infty)$ for the homogeneous system

$$\vec{x}\,' = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\vec{x}$$

use variation of parameters to help find the general solution of

$$\vec{x}\,' = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\vec{x} + \begin{bmatrix} 1/t \\ 1/t \end{bmatrix}$$

on $(0,\infty)$.

Step 1: $\vec{x}_c(t) = c_1\begin{bmatrix} t+1 \\ t \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step 2: $$\vec{x}_p(t) = \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\,\vec{g}(s)\,ds$$

$$\Psi^{-1}(s) = \frac{1}{\det \Psi(s)}\begin{bmatrix} \psi_{22}(s) & -\psi_{12}(s) \\ -\psi_{21}(s) & \psi_{11}(s) \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -s & s+1 \end{bmatrix}$$

$$\therefore\ \vec{x}_p(t) = \Psi(t)\int_{t_0}^{t}\begin{bmatrix} 1 & -1 \\ -s & s+1 \end{bmatrix}\begin{bmatrix} 1/s \\ 1/s \end{bmatrix}ds = \Psi(t)\int_{1}^{t}\begin{bmatrix} 0 \\ -1+1+\frac{1}{s} \end{bmatrix}ds$$

$$= \begin{bmatrix} t+1 & 1 \\ t & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \ln(t) \end{bmatrix} = \ln(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Step 3: $\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t)$

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix} t+1 \\ t \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \ln(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Similar to #12, p. 440

Ex 2 (Similar to #12, p. 440) Find the general solution of

$$\vec{x}\,' = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \vec{x} + \underbrace{\begin{bmatrix} \sec(t) \\ 0 \end{bmatrix}}_{\vec{g}(t)} \quad \text{on } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

Step 1: Solve $\vec{x}\,' = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{A}\vec{x}$. $\quad \vec{x}_c = \vec{k}e^{\lambda t}$ leads to $\lambda\vec{k} = A\vec{k}$.

Eigenvalues	Eigenvectors
$\lambda_1 = i$	$\vec{k}^{(1)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$
$\lambda_2 = -i$	$\vec{k}^{(2)} = \overline{\vec{k}^{(1)}}$

$$\vec{x}^{(1)} = \vec{k}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{it} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)(\cos(t) + i\sin(t))$$

$$\tilde{\vec{x}}^{(1)} = \mathrm{Re}\left(\vec{x}^{(1)}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

$$\tilde{\vec{x}}^{(2)} = \mathrm{Im}\left(\vec{x}^{(1)}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(t) = \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}$$

$$\therefore \; \vec{x}_c = c_1\tilde{\vec{x}}^{(1)} + c_2\tilde{\vec{x}}^{(2)} = c_1\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + c_2\begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}.$$

Step 2: $\vec{x}_p = \Psi(t)\displaystyle\int_{t_0}^{t} \Psi^{-1}(s)\,\vec{g}(s)\,ds$ where $\Psi(t) = \left[\tilde{\vec{x}}^{(1)} \; \tilde{\vec{x}}^{(2)}\right] = \begin{bmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{bmatrix}$,

and $\vec{g}(t) = \begin{bmatrix} \sec(t) \\ 0 \end{bmatrix}$.

$$\therefore \; \Psi^{-1}(t) = \frac{1}{\det \Psi(t)}\begin{bmatrix} \psi_{22} & -\psi_{12} \\ -\psi_{21} & \psi_{11} \end{bmatrix}$$

$$= \frac{1}{-1}\begin{bmatrix} -\cos(t) & -\sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{bmatrix}$$

Note: $\Psi^{-1}(t) = \Psi(t)$. Beware this is very unusual.

$$\Psi^{-1}(s)\ \overrightarrow{\ }(s) = \begin{bmatrix} \cos(s) & \sin(s) \\ \sin(s) & -\cos(s) \end{bmatrix} \begin{bmatrix} \sec(s) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \tan(s) \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{bmatrix} \int_0^t \begin{bmatrix} 1 \\ \tan(s) \end{bmatrix} ds = \begin{bmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{bmatrix} \begin{bmatrix} t \\ -\ln(\cos(t)) \end{bmatrix}$$

$$\vec{x}_p = t\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} - \ln(\cos(t))\begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}.$$

<u>Step 3</u>: The general solution on $(-\pi/2, \pi/2)$ is

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\boxed{\vec{x} = c_1\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + c_2\begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix} + t\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} - \ln(\cos(t))\begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}}.$$

<u>Alternate Examples</u>:

Ex 2 (Fall 2005 Final Exam) Use VoP to find a particular solution of $\vec{x}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t$, given that $\Phi(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}$ is a fundamental matrix of the associated homogeneous equation.

<u>Solution</u>: $\vec{x}_p = \begin{bmatrix} 4t+3 \\ 2t+3 \end{bmatrix}e^t$

Ex 3 Find the general solution of $\vec{x}' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}\vec{x} + \begin{bmatrix} -7 \\ 7 \end{bmatrix}$ given that $\Phi(t) = \begin{bmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{bmatrix}$ is a fundamental matrix for the associated homogeneous equation.

<u>Solution</u>: $\vec{x} = c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{7t} + \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ ← (Undetermined Coefficients)