Uniform Convergence and Differentiation

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of differentiable functions on $[a,b]$.

I. Suppose $f_n' \to g$ uniformly on $[a,b]$.

<u>Q</u>: Does $\{f_n\}_{n=1}^{\infty}$ converge to a differentiable function f such that $f' = g$ on $[a,b]$?

(I.e. is $\left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} f_n'$?)

<u>A</u>: No! Consider $f_n(x) = n$ on $[0,1]$. Then $f_n' = 0$ yet f_n does not converge.

Suppose in addition to I. that

natural in light of counterexample → II. $\{f_n(x_0)\}_{n=1}^{\infty}$ is a convergent sequence of real numbers for some $x_0 \in [a,b]$, and

appears pretty "heavy-handed" → III. f_n' is continuous on $[a,b]$ for $n = 1, 2, 3, \ldots$

By the F.T.C. (Thm 6.21)

$$f_n(x) - f_n(x_0) = \int_{x_0}^{x} f_n'(t)\,dt \qquad (x \in [a,b],\ n = 1, 2, 3, \ldots)$$

<u>Claim</u>: $f_n(x) - f_n(x_0) \longrightarrow \int_{x_0}^{x} g(t)\,dt$ uniformly on $[a,b]$.

$$\left| f_n(x) - f_n(x_0) - \int_{x_0}^{x} g(t)\,dt \right| = \left| \int_{x_0}^{x} \left(f_n'(t) - g(t)\right) dt \right|$$

$$\leq \int_{x_0}^{x} \left| f_n'(t) - g(t) \right| dt$$

$$\leq \underbrace{\|f_n' - g\|_u \cdot (b-a)}_{\text{This goes to } 0 \text{ as } n\to\infty, \text{ independent of } x \in [a,b].}$$

We have shown that $\{f_n\}$ converges uniformly on $[a,b]$ to

$$f(x) = y_0 + \int_{x_0}^{x} g(t)\,dt$$

where $y_0 = \lim_{n\to\infty} f_n(x_0)$. The F.T.C. (Thm 6.20) shows that $f' = g$ on $[a,b]$.

(~~Wade~~) Theorem 7.17. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\boxed{\cancel{\text{continuously}}}$ differentiable functions on $[a,b]$. If $f_n' \to g$ uniformly on $[a,b]$ and $\{f_n(x_0)\}_{n=1}^{\infty}$ converges for some $x_0 \in [a,b]$ then $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$ to a differentiable function f and $f' = g$.

Notes: The assumption that each f_n' is continuous can be deleted, but the proof is more technical. In this case:

(a) use the MVT to show $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy on $[a,b]$, and hence convergent, say to f;

(b) use Theorem 7.11 to show that f is differentiable and $f' = \lim_{n\to\infty} f_n'$.

See Rudin p. 152 for details.

Example: Let $\{b_n\}_{n=1}^{\infty}$ be a bounded real sequence. Show that

$$(*) \qquad u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}$$

is a solution of the diffusion equation $u_t - u_{xx} = 0$ in $H^+: -\infty < x < \infty,\ 0 < t < \infty$.

Recall: The series defining u in $(*)$ converges uniformly on each $H_\delta^+ = \{(x,t) \in \mathbb{R}^2 : t \geq \delta\}$ for $\delta > 0$ and u is a continuous function on H^+.

Solution: Fix $(x_0, t_0) \in H^+$ and choose $\delta > 0$ such that $0 < \delta < t_0$. For $N = 1, 2, 3, \ldots$ and $t \in [\delta, \infty)$, consider

$$f_N(t) = \sum_{n=1}^{N} b_n \sin(n\pi x_0) e^{-n^2\pi^2 t}.$$

Clearly $f_N'(t) = \sum_{n=1}^{N} -n^2\pi^2 b_n \sin(n\pi x_0) e^{-n^2\pi^2 t}$. For $n=1,2,3,\ldots$ let

$$M_n = \sup\left\{ \left| -n^2\pi^2 b_n \sin(n\pi x_0) e^{-n^2\pi^2 t} \right| : \ t \geq \delta \right\} \quad \left(\leq n^2\pi^2 b_n e^{-n^2\pi^2\delta} \right).$$

Then $\sum_{n=1}^{\infty} M_n \leq \pi^2 B \sum_{n=1}^{\infty} n^2 \left(e^{-\pi^2\delta}\right)^{n^2} \leq \pi^2 B \sum_{n=1}^{\infty} n^2 \gamma^n$ where $\gamma = e^{-\pi^2\delta} \in (0,1)$.

But if $0 < \gamma < 1$ then $\sum_{n=1}^{\infty} n^2\gamma^n$ is convergent (say, by the integral test : #8, pp. 138-9). Therefore, by the Weierstrass M-test

$$f_N'(t) \longrightarrow g(t) = \sum_{n=1}^{\infty} -n^2\pi^2 b_n \sin(n\pi x_0) e^{-n^2\pi^2 t} \quad \underline{\text{uniformly}}$$

in $[\delta, \infty)$. Clearly $\left\{ f_N(t_0) \right\}_{N=1}^{\infty} = \left\{ \sum_{n=1}^{N} b_n \sin(n\pi x_0) e^{-n^2\pi^2 t_0} \right\}_{N=1}^{\infty}$ converges.

By Theorem 7.17 $\{f_N\}$ converges uniformly on $\delta \leq t \leq b$ (arbitrary) to a differentiable function $f(t)$ $\left(= \sum_{n=1}^{\infty} b_n \sin(n\pi x_0) e^{-n^2\pi^2 t} \right)$ and $f' = g$, i.e. $\left(\sum_{n=1}^{\infty} b_n \sin(n\pi x_0) e^{-n^2\pi^2 t} \right)' = \sum_{n=1}^{\infty} -n^2\pi^2 b_n \sin(n\pi x_0) e^{-n^2\pi^2 t}$ for $\delta \leq t \leq b$.

We have shown that

$$\frac{\partial u}{\partial t}(x_0, t_0) = \sum_{n=1}^{\infty} -n^2\pi^2 b_n \sin(n\pi x_0) e^{-n^2\pi^2 t_0}.$$

Similar arguments show

$$\frac{\partial u}{\partial x}(x_0, t_0) = \sum_{n=1}^{\infty} n\pi b_n \cos(n\pi x_0) e^{-n^2\pi^2 t_0}$$

and

$$\frac{\partial^2 u}{\partial x^2}(x_0, t_0) = \sum_{n=1}^{\infty} -n^2\pi^2 b_n \sin(n\pi x_0) e^{-n^2\pi^2 t_0}.$$

Therefore $\frac{\partial u}{\partial t}(x_0, t_0) - \frac{\partial^2 u}{\partial x^2}(x_0, t_0) = 0$.

Uniform Convergence and Integration

Theorem 7.16: Let $\alpha \in BV[a,b]$ and $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ $(n=1,2,3,\ldots)$. If $f_n \to f$ uniformly on $[a,b]$ then $f \in \mathcal{R}(\alpha)$ and

$$\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha .$$

I.e.

$$\lim_{n\to\infty} \int_a^b f_n\,d\alpha = \int_a^b \left(\lim_{n\to\infty} f_n\right) d\alpha$$

Proof: See Rudin pp. 151–2.

Corollary: Let $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ $(n=1,2,3,\ldots)$. If

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in [a,b])$$

with the (partial sums of the) series converging uniformly on $[a,b]$, then

$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\,d\alpha .$$

Ex (#10 p.167) Consider $f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2}$ for $x \in \mathbb{R}$.

(a) Find all the discontinuities of f, and show that they form a countable, dense set in $\mathbb{R}$.

(b) Show never-the-less that f is Riemann integrable on every bounded interval.

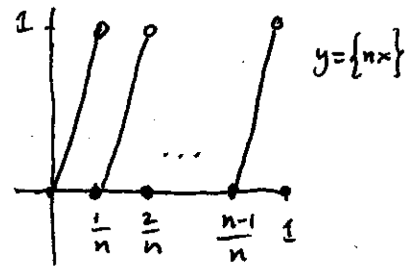
(a) Let $D(f)$ denote the set of points in $\mathbb{R}$ where f is

discontinuous. Previously we showed that $\mathcal{D}(f) \subseteq \mathbb{Q}$. A straightforward application of Theorem 7.11 shows that $\mathbb{Q} \subseteq \mathcal{D}(f)$ (see back side for details). cf. HW Set #5
Hence $\mathcal{D}(f) = \mathbb{Q}$, a countable dense subset of $\mathbb{R}$.

(b) Since f is periodic with period 1, it suffices to show that $f \in \mathcal{R}$ on $[0,1]$.

<u>Recall</u>: The sequence $\left\{ \sum_{n=1}^{N} \frac{\{nx\}}{n^2} \right\}_{N=1}^{\infty}$ of partial sums of the series defining f converges uniformly to f on $\mathbb{R}$ by the Weierstrass M-test.

Because $f_n(x) = \dfrac{\{nx\}}{n^2}$ $(n = 1, 2, 3, \dots)$ has only finitely many discontinuities in $[0,1]$, each $f_n \in \mathcal{R}$ on $[0,1]$. The Corollary to Theorem 7.16 then implies $f \in \mathcal{R}$ on $[0,1]$ and



$$\int_0^1 f\,dx = \sum_{n=1}^{\infty} \int_0^1 \frac{\{nx\}}{n^2}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 \{nx\}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{2n^2} \quad \leftarrow \tfrac{1}{2}\zeta(2)$$

$$= \frac{\pi^2}{12}.$$

The Banach Space $C[a,b]$.

Real Rudin, pp. 154 (bottom) – 165.

Def: A (real) <u>normed linear space</u> is a (real) vector space X together with a function $\|\cdot\|: X \to \mathbb{R}$ with the following properties:

(1) $\|x\| \geq 0$ for all $x \in X$, with equality only if $x = 0$;

(2) $\|cx\| = |c| \cdot \|x\|$ for all $x \in X$ and all $c \in \mathbb{R}$;

(3) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

<u>Examples of normed linear spaces</u>:

1. Let $X = \mathbb{R}^n$ and define $\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

   (1) clear

   (2) clear

   (3) Thm 1.37(e)

2. Let $X = C[a,b]$ (the vector space of all continuous <u>real</u>-valued functions with domain $[a,b]$) and define

$$\|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{1/2} \quad \text{for} \quad f \in C[a,b].$$

   (1) See #2, p. 138.

   (2) clear.

   (3) See #11, p. 140

   (Could replace 2 by any $p \in [1, \infty)$ and we would still have a norm on $C[a,b]$.)

3. Let $X = C[a,b]$ and define $\|f\|_u = \sup \{ |f(x)| : x \in [a,b] \}$ for all $f \in C[a,b]$.

   (1) clear.

   (2) clear.

   (3) Let $f, g \in C[a,b]$ and $x \in [a,b]$. Then

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_u + \|g\|_u.$$

   Taking the supremum of the LHS as x varies over $[a,b]$, we obtain

$$\|f+g\|_u \leq \|f\|_u + \|g\|_u.$$

Note: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $C[a,b]$. Using Theorem 7.12, it is easy to see that $f_n \to f$ uniformly on $[a,b]$ if and only if $f \in C[a,b]$ and $\|f - f_n\|_u \to 0$ as $n \to \infty$. (This can be generalized.)

Def: Let (X, d) be a metric space and denote by $C(X)$ the real vector space of all bounded, real-valued, continuous functions with domain X. Define the uniform norm on $C(X)$ by

$$\|f\|_u = \sup\{|f(x)| : x \in X\} \qquad (f \in C(X)).$$

Note: It is routine to check that $(C(X), \|\cdot\|_u)$ is a normed linear space.

Theorem 7.15: Let (X, d) be a metric space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $C(X)$. The following are equivalent:

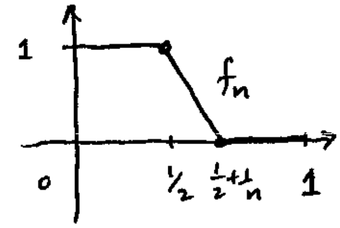
(a) $\|f_n - f_m\|_u \to 0$ as $m, n \to \infty$.

(b) There exists $f \in C(X)$ such that $\|f - f_n\|_u \to 0$ as $n \to \infty$.

Proof: (b) $\Rightarrow$ (a) clear.

(a) $\Rightarrow$ (b) Apply Theorems 7.8 and 7.12. (See Rudin p. 151 for details)

Notes: A sequence $\{f_n\}$ satisfying (a) is called a Cauchy sequence in $(C(X), \|\cdot\|_u)$. A sequence $\{f_n\}$ satisfying (b) is called a convergent sequence in $(C(X), \|\cdot\|_u)$. Although convergent sequences are Cauchy sequences, the converse does not hold in all normed linear spaces. A normed linear space in which every Cauchy sequence is convergent is called either complete or a Banach space (Stefan Banach, 1892–1945). In particular $(C[a,b], \|\cdot\|_u)$ is a Banach space. An example of a normed linear space which is not a Banach space is $(C[a,b], \|\cdot\|_2)$. For instance, consider the sequence $f_n : [0,1] \to \mathbb{R}$ $(n=1,2,3,\dots)$ whose graphs are:



Clearly each $f_n \in C[0,1]$ and, if $1 \le m < n$, then

$$\|f_m - f_n\|_2^2 = \int_0^1 |f_m(x) - f_n(x)|^2 dx = \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} |f_m(x) - f_n(x)|^2 dx \le \frac{1}{m},$$

so $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(C[0,1], \|\cdot\|_2)$. However,

if $g(x) = \begin{cases} 1/2 & \text{if } 0 \le x \le 1/2, \\ 0 & \text{if } 1/2 < x \le 1, \end{cases}$ then $\|g - f_n\|_2^2 = \int_{1/2}^{\frac{1}{2}+\frac{1}{n}} f_n^2(x)\,dx \le \frac{1}{n}$ $(n=1,2,3,\dots)$

so there is no continuous function $f : [0,1] \to \mathbb{R}$ such that $\|f - f_n\|_2 \to 0$.

We will see later that the completion of $(C[a,b], \|\cdot\|_2)$ (cf. #24, p. 82) is a Banach space, namely the Lebesgue space $L^2[a,b]$ (cf. p. 325f).

<u>Uniform Approximation in $C[a,b]$</u> (Read pp. 159-165 in Rudin)

<u>Theorem 7.26</u> (Weierstrass)

Let $f \in C[a,b]$. Then there exists a sequence of polynomial functions $\{P_n\}_{n=1}^{\infty}$ on $[a,b]$ such that $P_n \to f$ uniformly on $[a,b]$.

<u>Proof</u>: By rescaling, we can assume that $[a,b] = [0,1]$.

We present a proof due to Serge Bernstein (1912). Fix $f \in C[0,1]$. Define the n^{th} Bernstein polynomial for f on $[0,1]$ by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $0 \le k \le n$.

<u>Claim</u>: To each $\varepsilon > 0$ there corresponds an integer $N = N(\varepsilon)$ such that

$$|f(x) - B_n(f;x)| < \varepsilon$$

for all $x \in [0,1]$ and all $n \ge N$.

In order to prove the claim, we will make use of three special cases:

$[f(x)=1]$ 1. $B_n(1;x) = 1$ for all $x \in [0,1]$ and all $n \ge 1$.

$[f(x)=x]$ 2. $B_n(x;x) = x$ " " " " " ".

$[f(x)=x^2]$ 3. $B_n(x^2;x) = x^2 + \frac{x(1-x)}{n}$ for all $x \in [0,1]$ and all $n \ge 1$.

To establish 1, 2, and 3, first observe that

$$(\dagger) \qquad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad (n = 1,2,3,\ldots,\ x,y \in \mathbb{R}).$$

1. $B_n(1;x) = \sum_{k=0}^{n} \binom{n}{k} x^k \underbrace{(1-x)}_{y}^{n-k} = (x + (1-x))^n = 1 \qquad (x \in \mathbb{R}, n=1,2,3,\ldots)$

2. Differentiate $(\dagger)$ w.r.t. x and multiply both sides by $\frac{x}{n}$:

$$(\dagger\dagger) \qquad x(x+y)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} \frac{k}{n} x^k y^{n-k}$$

Now set $y = 1-x$ in (††) to obtain

$$x = \sum_{k=1}^{n} \binom{n}{k} \frac{k}{n} x^{k}(1-x)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^{k}(1-x)^{n-k} = B_n(x;x)$$

for $n = 1, 2, 3, \ldots$ and all $x \in \mathbb{R}$.

3. Differentiate (††) w.r.t. x and then multiply both sides by $\frac{x}{n}$:

$$\frac{x}{n}(x+y)^{n-1} + \left(\frac{n-1}{n}\right)x^{2}(x+y)^{n-2} = \sum_{k=1}^{n} \binom{n}{k}\left(\frac{k}{n}\right)^{2} x^{k} y^{n-k}$$

$$(†††) \qquad x^{2}(x+y)^{n-2} + \frac{x}{n}(x+y)^{n-2}\left[\cancel{x}+y-\cancel{x}\right] = \sum_{k=1}^{n} \binom{n}{k}\left(\frac{k}{n}\right)^{2} x^{k} y^{n-k}$$

Now set $y = 1-x$ in (†††):

$$x^{2} + \frac{x(1-x)}{n} = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{k}{n}\right)^{2} x^{k}(1-x)^{n-k} = B_n(x^{2}; x)$$

<u>Proof of Claim</u>: Let $\varepsilon > 0$.

Uniform continuity of f on $[0,1]$ implies the existence of $\delta = \delta(\varepsilon) > 0$ such that

$$\left|f(x) - f\left(\frac{k}{n}\right)\right| < \frac{\varepsilon}{2} \quad \text{if} \quad \left|x - \frac{k}{n}\right| < \delta.$$

Fix $x \in [0,1]$ and let $n \geq 1$. Then

$$|f(x) - B_n(f;x)| = \left| f(x) - \sum_{k=0}^{n} \binom{n}{k} f\left(\tfrac{k}{n}\right) x^k (1-x)^{n-k} \right|$$

$$= \left| \sum_{k=0}^{n} \binom{n}{k} \left[f(x) - f\left(\tfrac{k}{n}\right)\right] x^k (1-x)^{n-k} \right|$$

$$\leq \sum_{\left|\frac{k}{n}-x\right|<\delta} \binom{n}{k} \left| f(x) - f\left(\tfrac{k}{n}\right)\right| x^k (1-x)^{n-k}$$

$$+ \sum_{\left|\frac{k}{n}-x\right|\geq\delta} \binom{n}{k} \left| f(x) - f\left(\tfrac{k}{n}\right)\right| x^k (1-x)^{n-k}$$

$$\equiv S_1(x) + S_2(x) .$$

We estimate $S_1(x)$ and $S_2(x)$ as follows:

$$0 \leq S_1(x) < \sum_{\left|\frac{k}{n}-x\right|<\delta} \binom{n}{k} \frac{\varepsilon}{2} x^k (1-x)^{n-k} \leq \frac{\varepsilon}{2} \overbrace{\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}}^{B_n(1;x)} = \frac{\varepsilon}{2} .$$

Let $M = \sup\{|f(x)| : x \in [0,1]\}$.

$$0 \leq S_2(x) \leq \sum_{\left|\frac{k}{n}-x\right|\geq\delta} \binom{n}{k} 2M x^k (1-x)^{n-k} \leq \frac{2M}{\delta^2} \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{k}{n}\right)^2 x^k (1-x)^{n-k}$$

$$= \frac{2M}{\delta^2}\left[x^2 \overbrace{B_n(1;x)}^{1} - 2x \overbrace{B_n(x;x)}^{x} + \overbrace{B_n(x^2;x)}^{x^2 + \frac{x(1-x)}{n}} \right]$$

$$= \frac{2M\, x(1-x)}{\delta^2 \cdot n}$$

$$\leq \frac{M}{2\delta^2 n}$$

$$< \frac{\varepsilon}{2} \quad \left(\textit{provided we choose } n > \frac{M}{\delta^2 \varepsilon} \right).$$

$$\therefore \quad |f(x) - B_n(f;x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \textit{for all } x \in [0,1] \textit{ and all } n > \frac{M}{\delta^2 \varepsilon} .$$

Q.E.D.

Comments on Bernstein's proof of the Weierstrass Approximation Theorem from

Methods of Real Analysis

by R.R. Goldberg.

10.2E. The theory of probability throws some light on the preceding proof. Suppose a coin is tossed and that the probability of heads is x while the probability of tails is, accordingly, $1-x$. If the coin is tossed n times, then the probability of exactly k heads in the n tosses is $\binom{n}{k}x^k(1-x)^{n-k}$. [This expression occurs in the definition of $B_n(x)$.]

The expected number of heads in n tosses is nx. (This is a technical probabilistic fact that is surely believable even to those who do not know the precise definition of "expected number.") Indeed, one feels sure that one is more likely to obtain precisely k heads in n tosses for the value (or values) of k close to nx than for those k that are far from nx. Thus Σ' refers to the k for which precisely k heads in n tosses is "more probable," while Σ'' refers to the k for which precisely k heads in n tosses is "less probable."

Indeed, the proof of 10.2A that we have given is essentially the same as one of the more familiar proofs of the "weak law of large numbers."

$$\Sigma' \longleftrightarrow S_1(x)$$

$$\Sigma'' \longleftrightarrow S_2(x)$$