Mathematics 315 Final Exam, Part I Name:____________________

Spring 2014

This portion of the 200 point final examination is "closed book". You are to turn in your solutions to the problems on this portion before receiving the second part. I recommend that you spend no more than 60 minutes on this portion of the exam.

1.(30 pts.) (a) State Littlewood's Three Principles.

(b) State a rigorous version for each one of Littlewood's principles.

2.(30 pts.) (a) State Lebesgue's Monotone Convergence Theorem.

(b) Give an example to show that the conclusion of the Monotone Convergence Theorem need not hold for sequences $\langle f_n \rangle_{n=1}^{\infty}$ of measurable functions satisfying $f_n(x) \geq f_{n+1}(x) \geq 0$ for all x in E and all $n \geq 1$.

(c) State Fatou's Lemma.

(d) Give an example to show that strict inequality may occur in Fatou's Lemma.

(e) State Lebesgue's Dominated Convergence Theorem.

(f) Use Fatou's Lemma to prove Lebesgue's Dominated Convergence Theorem.

3.(40 pts.) In each of the following, compute the Lebesgue integral of f over the set E or show that f is not integrable over E. The symbol $\mathbb{A}$ represents the set of algebraic numbers and P denotes the Cantor ternary set. Please justify all steps in your computations.

(a) $$f(x) = \begin{cases} -1 & \text{if } x \in P, \\ 2 & \text{if } x \in [0,1] \setminus P, \\ 3 & \text{if } x \in [-1,0] \setminus (-P), \\ -4 & \text{if } x \in [-1,0] \cap (-P). \end{cases} \qquad E = [-1,1]$$

(b) $$f(x) = \begin{cases} \sin(x) & \text{if } x \in \mathbb{A}, \\ \dfrac{1}{\sqrt[3]{x}} & \text{if } x \in [0,1] \setminus \mathbb{A}. \end{cases} \qquad E = [0,1]$$

(c) $$f(x) = \frac{\sin(x)}{x} \qquad E = (0,\infty)$$

(d) $$f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \qquad E = [0,\pi]$$

Mathematics 315 Final Exam, Part II Name:___________________
Spring 2014 (1 pt.)

This portion of the 200 point final examination is "open book". That is, you may freely use your two textbooks for this class, Rudin's *Principles of Mathematical Analysis* and Royden and Fitzpatrick's *Real Analysis*. **Work any THREE problems of your choosing.** Please **CIRCLE** the numbers of the problems on this portion whose solutions you wish me to grade. All problems have the same point value, 33 points.

GROUP A

1. Let H denote the vector space of continuous functions f on the closed interval $[0,1]$ satisfying $f(0)=f(1)$ and $\sum_{n=-\infty}^{\infty}(1+n^2)|\hat{f}(n)|^2<\infty$. (Here $\hat{f}(n)=\int_0^1 f(x)e^{-2\pi inx}dx \quad (n=0,\pm1,\pm2,...)$ denote the Fourier coefficients of f.)

(a) Show that $\|f\|_1=\left(\sum_{n=-\infty}^{\infty}(1+n^2)|\hat{f}(n)|^2\right)^{1/2}$ defines a norm on H.

(b) Show that there exists a constant $C>0$ such that $\|f\|_u\le C\|f\|_1$ for all f in H.

(Here $\|f\|_u=\sup\{|f(x)|:0\le x\le1\}$ denotes the uniform norm of f.)

2. Let f and g be continuous functions on the closed interval $[0,1]$ and let α be of bounded variation on $[0,1]$. Define

$$\beta(x)=\int_0^x f d\alpha \quad (0\le x\le1).$$

Show that β is of bounded variation on $[0,1]$ and $\int_0^1 g d\beta=\int_0^1 gf d\alpha$. (Caution: The fundamental theorem of calculus cannot be used to differentiate β because the integral defining it is a Stieltjes integral with respect to a function α of bounded variation on $[0,1]$.)

GROUP B

3. Let X be the subset of the closed interval $[0,1]$ consisting of those real numbers x such that, in the decimal representation of x, the first appearance of the digit 2 precedes the first appearance of the digit 3. For example, $\pi-3=.141592653...$ belongs to X while $13/40=.325$ does not belong to X. Show that X is Lebesgue measurable and find its measure.

4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on $[0,1]$ with each

$$\int_0^1 |f_n|^2\, dm\le10$$

and $f_n(x)\to0$ a.e. on $[0,1]$. Show that

$$\int_0^1 |f_n|\,dm\to0.$$

(Hint: Egoroff and Cauchy-Schwarz may be useful.)