Mathematics 315
Introduction to Mathematical Analysis
Qualifying Examination
Fall 2006

Name: ________________________

This is a two hour examination in which you may refer at any time to your textbooks for Math 315: Principles of Mathematical Analysis by Walter Rudin and Real Analysis by H. L. Royden. However, no other aids (books, lecture notes, homework solutions, exam solutions, calculators, etc.) are permitted.

This examination consists of 8 problems of equal value, grouped into two parts. You are to solve **5 problems** of your choosing, subject to the constraint that **at least two problems must be chosen from Part I and at least two problems must be chosen from Part II**. The minimum score for a passing grade will be 70 percent.

PART I

1. Let $\alpha(x)$ denote the fractional part of the real number x. For example,

$$\alpha(5/4) = .25, \ \alpha(2) = 0, \text{ and } \alpha(\pi) = .1415926...$$

(a) Compute the total variation of α on the interval [1,4].

(b) Show that the product of two functions of bounded variation on a closed bounded interval is of bounded variation on that interval.

(c) Let $f(x) = 1/x$ and $\beta(x) = \alpha^2(x)$. Why is f Riemann-Stieltjes integrable with respect to β on the interval [1,4]?

(d) Evaluate the Riemann-Stieltjes integral of f with respect to β on the interval [1,4].

2. (a) If $k \in \mathbb{Z}$ and $f(x) = e^{ikx}$, show that

$$(*) \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

(b) Show that (*) holds for every complex, continuous, $2\pi-$periodic function f on $\mathbb{R}$.

(c) Does (*) hold for every complex, bounded, measurable, $2\pi-$periodic function f on $\mathbb{R}$? Prove your assertion.

3. Let f be the $2\pi-$periodic function defined on a fundamental period by the formula

$$f(x) = x^2 - \frac{\pi^2}{3} \quad \text{if } -\pi \le x < \pi.$$

Show, by rigorous argument, that

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)e^{-n^2 t}$$

defines a function which solves the diffusion equation $u_t = u_{xx}$ in the region $t > 0$ of the $xt-$plane and which satisfies the initial condition $u(x,0) = f(x)$ for $-\infty < x < \infty$.

4. Let F be a continuous real function in the closed unit cube

$$Q = \{(x,y,z) : 0 \le x \le 1, \ 0 \le y \le 1, \ 0 \le z \le 1\}$$

in $\mathbf{R}^3$. Show that to each positive number ε there corresponds a positive integer N and a (finite) collection f_k, g_k, h_k $(1 \le k \le N)$ of real polynomials in the interval [0,1] such that

$$\left| F(x,y,z) - \sum_{k=1}^{N} f_k(x)g_k(y)h_k(z) \right| < \varepsilon$$

for all (x,y,z) in Q.

PART II

5. In the following problem, P denotes the Cantor ternary set, $\mathbb{Q}$ denotes the set of rational numbers, and $\mathbb{A}$ denotes the set of algebraic numbers. (Recall that a real number r is algebraic if there exists a nonzero polynomial P with integer coefficients such that $P(r) = 0$.) In each case, compute the Lebesgue integral of f over the set E, or show that f is not integrable over E. Please justify the steps in your computations.

(a) $f(x) = \begin{cases} 3 & \text{if } x \in P, \\ -1 & \text{if } x \in [0,1] \setminus P, \\ 2 & \text{if } x \in [-1,0] \setminus \mathbb{Q}, \\ -4 & \text{if } x \in [-1,0] \cap \mathbb{Q}. \end{cases}$ $\quad E = [-1,1]$.

(b) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{A}, \\ \dfrac{1}{x} & \text{if } x \in \mathbb{R} \setminus \mathbb{A}. \end{cases}$ $\quad E = [0,1]$.

(c) $f(x) = \begin{cases} e^{x^2} & \text{if } x \in \mathbb{Q}, \\ \cos(x)e^{-x} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ $\quad E = (0, \infty)$.

6. Let E denote the set of real numbers in the interval $[0,1]$ which possess a decimal expansion which contains no 2's and no 7's. For example, the numbers $1/2 = .5$ and $7/10 = .6999\ldots$ belong to E, while the numbers $1/4 = .25$ and $1/\sqrt{2} = .7071\ldots$ do not.

(a) Compute the Lebesgue measure of E.

(b) Determine, with proof, whether or not E is a Borel set.

7. Let f be the function defined on the interval $[0,1]$ as follows: $f(x) = 0$ if x is a point of the Cantor ternary set and $f(x) = 1/k$ if x is in one of the complementary open intervals of the Cantor set with length 3^{-k}. For example, $f(1/3) = 0$, $f(1/2) = 1$, and $f(4/5) = 1/2$.

(a) Show that f is a Lebesgue measurable function.

(b) Evaluate $\int_0^1 f(x)dx$.

8. Let f be a function defined and bounded on the unit square

$$S = \{(x,t) : 0 < x < 1,\ 0 < t < 1\}.$$

Suppose that:

(a) for each fixed t in $(0,1)$ the function $x \mapsto f(x,t)$ is measurable,

(b) at each (x,t) in S, the partial derivative $\dfrac{\partial f}{\partial t}$ exists, and

(c) $\dfrac{\partial f}{\partial t}$ is a bounded function in S.

Show that $\dfrac{d}{dt}\displaystyle\int_0^1 f(x,t)dx = \int_0^1 \frac{\partial f}{\partial t}(x,t)dx.$