Mathematics 315

Introduction to Mathematical Analysis

Qualifying Examination

Fall 2007

Name: ___________________________________

This is a two hour examination in which you may refer at any time to your textbooks for Math 315: Principles of Mathematical Analysis by Walter Rudin and Real Analysis by H. L. Royden. However, no other aids (books, lecture notes, homework solutions, exam solutions, calculators, etc.) are permitted.

This examination consists of 8 problems of equal value, grouped into two parts. You are to solve **5 problems** of your choosing, subject to the constraint that **at least two problems must be chosen from Part I and at least two problems must be chosen from Part II**. The minimum score for a passing grade will be 70 percent.

PART I

1. Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \ldots$ denote the (infinite) sequence of prime numbers, let $\pi(x)$ denote the number of primes less than or equal to x, and let $f(x) = \dfrac{1}{x}$ for $x > 0$.

(a) Show that $\displaystyle\int_1^x f\,d\pi = \sum_{p_k \le x} \frac{1}{p_k}$ for $x > 2$.

(b) Show that $\displaystyle\int_1^x f\,d\pi = \frac{\pi(x)}{x} + \int_1^x \frac{\pi(t)}{t^2}\,dt$ for $x > 2$.

(c) Use (a) and (b) to verify that

$$\sum_{p_k \le x} \frac{1}{p_k} = \ln(\ln(x)) + \int_e^x \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2} + \int_1^e \frac{\pi(t)}{t^2}\,dt + \frac{\pi(x)}{x} \qquad \text{for } x > 2.$$

In the remainder of this problem you may assume that to each $a > 0$ there correspond real constants $B = B(a) > 1$ and $C = C(a) > 0$ such that

(*) $$\left|\pi(x) - \frac{x}{\ln(x)}\right| \le Cxe^{-a\sqrt{\ln(x)}} \quad \text{for } x \ge B.$$

(d) Use (*) to help show that $\displaystyle\lim_{y > x \to \infty} \left|\int_x^y \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2}\right| = 0.$

(e) Why does the improper Riemann integral $\displaystyle\int_e^\infty \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2}$ converge?

(f) Use (c) and (*) to help show that

$$\lim_{x \to \infty}\left(\sum_{p_k \le x} \frac{1}{p_k} - \ln(\ln(x))\right) = \int_e^\infty \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2} + \int_1^e \frac{\pi(t)}{t^2}\,dt.$$

2. (a) If $k \in \mathbb{Z}$ and $f(x) = e^{ikx}$, show that

(*) $$\lim_{N \to \infty} \frac{1}{N}\sum_{n=1}^{N} f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

(b) Show that (*) holds for every complex, continuous, 2π-periodic function f on $\mathbb{R}$.

(c) Does (*) hold for every complex, bounded, measurable, 2π-periodic function f on $\mathbb{R}$? Prove your assertion.

3. (a) Determine, with proof, which of the following functions define norms on the space BV[0,1] of functions of bounded variation on the interval [0,1].

$$N_1(f) = T(f;0,1)$$
$$N_2(f) = |f(0)| + T(f;0,1)$$
$$N_3(f) = \int_0^1 |f(x)|\,dx$$
$$N_4(f) = \sup\{|f(x)| : 0 \le x \le 1\}$$

(b) Determine, with proof, which of the norms N on BV[0,1] from part (a) of this problem make the normed linear space (BV[0,1], N) a Banach space.

4. Let f be the 2π-periodic function determined by $f(0)=0$ and

$$f(x)=\frac{\pi-x}{2} \quad \text{for} \quad 0<x<2\pi.$$

Show, by rigorous argument, that

$$u(x,t)=\sum_{n=1}^{\infty}\frac{\sin(nx)e^{-n^2t}}{n}$$

defines a function which solves the diffusion equation $u_t-u_{xx}=0$ in the upper halfplane $t>0$ of the xt-plane, and satisfies the initial condition $u(x,0)=f(x)$ for $-\infty<x<\infty$.

PART II

5. Let $\langle a_n \rangle_{n=1}^{\infty}$ be a positive divergent sequence, and for every positive integer n let

$$f_n(x)=\begin{cases} a_n & \text{if } x\in\left(\frac{1}{n+1},\frac{1}{n}\right), \\ 0 & \text{otherwise in } (0,1). \end{cases}$$

(a) If $\left\langle \frac{a_n}{n^2} \right\rangle_{n=1}^{\infty}$ is a bounded sequence, show that $\left\langle \int_0^1 f_n dx \right\rangle_{n=1}^{\infty}$ is a bounded sequence.

(b) Place an X in each blank below that would imply

$$\lim_{n\to\infty}\int_0^1 f_n dx=\int_0^1 \lim_{n\to\infty} f_n dx$$

and an O in each blank otherwise. Supply reasons for your answers.

(i) ________ $\left\langle \frac{a_n}{\ln(n)} \right\rangle_{n=2}^{\infty}$ is a bounded sequence.

(ii) ________ $\lim_{n\to\infty}\frac{a_n}{n^3\ln\left(1+\frac{1}{\sqrt{n}}\right)}=0.$

(iii) ________ $\lim_{n\to\infty}\frac{a_n}{n^2}=0.$

(iv) ________ $\left\langle \frac{a_n}{n^2\ln(n)} \right\rangle_{n=2}^{\infty}$ is a bounded sequence.

6. Let E denote the set of real numbers in the interval $[0,1]$ which possess a decimal expansion which contains no 2's and no 7's. For example, the numbers $1/2=.5$ and $7/10=.6999\ldots$ belong to E, while the numbers $1/4=.25$ and $1/\sqrt{2}=.7071\ldots$ do not.

(a) Compute the Lebesgue measure of E.

(b) Determine, with proof, whether or not E is a Borel set.

7. In this problem you may assume the following theorem: If $f \in L^1(-\pi,\pi)$ then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0 = \lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

Let $\langle n_k \rangle_{k=1}^{\infty}$ be an increasing sequence of positive integers, let E be the set of all x in $(-\pi,\pi)$ for which $\langle \sin(n_k x) \rangle_{k=1}^{\infty}$ is a convergent sequence, and let A be any measurable subset of E.

(a) Show that $\lim_{k\to\infty} \int_A \sin(n_k x)dx = 0.$

(b) Show that $\lim_{k\to\infty} 2\int_A (\sin(n_k x))^2\, dx = \lim_{k\to\infty} \int_A (1-\cos(2n_k x))\, dx = m(A).$

(c) Use (a) and (b) to help show that $m(E) = 0.$

8. Let f be a bounded measurable function on $[0,1]$ and define

$$F(x) = \int_0^x f(t)dt \quad \text{for } x \text{ in } [0,1].$$

(a) Show that F is continuous on $[0,1]$.

(b) Show that F is of bounded variation on $[0,1]$.

In the rest of this problem you may assume the following theorem: If g is increasing on (a,b) then $g'(x)$ exists a.e. in (a,b).

(c) Why does $F'(x)$ exist a.e. in $(0,1)$?

(d) Use (c) to help show that $\int_0^y \{F'(t) - f(t)\}\, dt = 0$ for all y in $[0,1]$.

(e) Use (c) and (d) to help show that $F'(x) = f(x)$ a.e. in $[0,1]$.