Mathematics 315
Introduction to Mathematical Analysis
Qualifying Examination
August 2012

This is a three hour examination in which you may refer at any time to your textbooks for Math 315: Principles of Mathematical Analysis by Walter Rudin, Real Analysis by H. L. Royden, and Introduction to Analysis by W. R. Wade. However, all other aids - books, lecture notes, homework and exam solutions, calculators, smart phones, etc. - are **NOT** permitted.

This examination consists of six problems of equal value arranged in two groups. You are to solve **FOUR** problems of your choosing, subject to the constraint that **two problems must be chosen from Group A and two problems must be chosen from Group B**. The minimum score for a passing grade on this exam is 70 percent.

GROUP A

1. (a) Let $\alpha(x)$ denote the fractional part of the real number x. For example, $\alpha(5/4) = .25$, $\alpha(2) = 0$, and $\alpha(\pi) = .1415926...$ Let $f(x) = 1/x$ and $\beta(x) = (\alpha(x))^2$. Why does the Riemann-Stieltjes integral of f with respect to β exist on the interval [1,4]?

(b) Evaluate the Riemann-Stieltjes integral of f with respect to β on the interval [1,4].

2. Let F be a continuous real function in the closed unit cube

$$Q = \{(x, y, z): 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}.$$

Show that to each positive number ε there corresponds a positive integer N and a (finite) collection f_k, g_k, h_k $(1 \le k \le N)$ of real polynomials on the interval [0,1] such that

$$\left| F(x, y, z) - \sum_{k=1}^{N} f_k(x) g_k(y) h_k(z) \right| < \varepsilon$$

for all (x, y, z) in Q.

3. Let f be the odd, 2π-periodic function determined by $f(x) = x(\pi - x)$ if $0 \le x \le \pi$. Show, by rigorous argument, that

$$u(x,t) = \sum_{k=0}^{\infty} \frac{8 \sin\left((2k+1)x\right) e^{-(2k+1)^2 t}}{\pi (2k+1)^3}$$

defines a function which solves the diffusion equation

$$u_t(x,t) - u_{xx}(x,t) = 0$$

at all points in the upper half-plane $\{(x,t):\ -\infty < x < \infty,\ 0 < t < \infty\}$ and which satisfies the initial condition $u(x,0) = f(x)$ for $-\infty < x < \infty$.

GROUP B

4. Let E denote the set of real numbers in the interval $[0,1]$ which possess a decimal expansion containing no even digits: 0, 2, 4, 6, 8. For instance, the numbers $1/9 = .111...$ and $1/5 = .1999...$ belong to E while the numbers $4/9 = .444...$ and $1/\sqrt{2} = .7071...$ do not.

(a) Compute the Lebesgue measure of E.

(b) Determine, with proof, whether E is a Borel set.

5. (a) Let f be Lebesgue integrable on $[0,1]$. Show that

$$m\left(\{x \in [0,1] : |f(x)| \ge \lambda\}\right) \le \frac{\|f\|_{L^1}}{\lambda} \quad \text{for all } \lambda > 0.$$

(b) Let f be a measurable function on $[0,1]$ with the property that there is a positive number C such that

$$m\left(\left\{x \in [0,1] : |f(x)| \geq \lambda\right\}\right) \leq \frac{C}{\lambda} \quad \text{for all } \lambda > 0.$$

Is it true that $f \in L^1[0,1]$? Justify completely your answer.

6. Let f be Lebesgue integrable on $[0,1]$ and define $F(x) = \int_0^x f(t)\,dt$ if $0 \leq x \leq 1$.

(a) Show that F is of bounded variation on $[0,1]$.

(b) Show that F is a continuous function on $[0,1]$.

(c) Derive a simple formula in terms of f for the value of the Riemann-Stieltjes integral $\int_0^1 x^2 dF$. If you use any results from the theory of Lebesgue measure and integration which were not covered in Math 315 but which you suspect hold, please note this clearly at the points where they appear in your calculations.