Solution to #4 on page 165 in Rudin.

#4. Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$

(a) For what values of x does the series converge absolutely?

(b) On what intervals does it converge uniformly?

(c) On what intervals does it fail to converge uniformly?

(d) Is f continuous wherever the series converges?

(e) Is f bounded?

(a) If $x > 0$ then

$$\sum_{n=1}^{\infty} \left|\frac{1}{1+n^2x}\right| = \sum_{n=1}^{\infty} \frac{1}{1+n^2x} < \sum_{n=1}^{\infty} \frac{1}{n^2x} = \frac{\pi^2}{6x} < \infty.$$

Suppose $x < 0$ and $x \neq -\frac{1}{n^2}$ for all $n = 1, 2, 3, \ldots$ Let n_x be the unique positive integer n satisfying $-\frac{1}{(n-1)^2} < x < -\frac{1}{n^2}$. Observe that

$$\left|\frac{1}{1+n^2x}\right| = \begin{cases} \dfrac{1}{1+n^2x} & \text{if } 1+n^2x > 0, \\[2ex] \dfrac{1}{-(1+n^2x)} & \text{if } 1+n^2x < 0, \end{cases}$$

$$= \begin{cases} \dfrac{1}{1+n^2x} & \text{if } x > -\frac{1}{n^2}, \\[2ex] \dfrac{1}{-1+n^2(-x)} & \text{if } x < -\frac{1}{n^2}, \end{cases}$$

so
$$\sum_{n=1}^{\infty} \left|\frac{1}{1+n^2x}\right| = \sum_{n=1}^{n_x-1} \frac{1}{1+n^2x} + \frac{1}{-1+n_x^2(-x)} + \sum_{n=n_x+1}^{\infty} \frac{1}{-1+n^2(-x)} \quad (*).$$

If $n \geq n_x+1$ then $x < -\frac{1}{n_x^2} < -\frac{1}{n^2}$ implies $n^2(-x) > \left(\frac{n}{n_x}\right)^2 > 1$,

so $-1+n^2(-x) > -1+\left(\frac{n}{n_x}\right)^2 > 0$ and hence $\dfrac{1}{-1+\left(\frac{n}{n_x}\right)^2} > \dfrac{1}{-1+n^2(-x)}$.

Therefore
$$\sum_{n=n_x+1}^{\infty} \frac{1}{-1+n^2(-x)} < \sum_{n=n_x+1}^{\infty} \frac{1}{-1+\left(\frac{n}{n_x}\right)^2} = n_x^2 \sum_{n=n_x+1}^{\infty} \frac{1}{n^2-n_x^2} =$$

Let $k = n - n_x$

$$= n_x^2 \sum_{n=n_x+1}^{\infty} \frac{1}{(n-n_x)(n+n_x)} = n_x^2 \sum_{k=1}^{\infty} \frac{1}{k(k+2n_x)} < n_x^2 \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2 n_x^2}{6} < \infty .$$

Substituting this estimate in (*) yields

$$\sum_{n=1}^{\infty} \left| \frac{1}{1+n^2 x} \right| < \sum_{n=1}^{n_x - 1} \frac{1}{1+n^2 x} + \frac{1}{-1+n_x^2(-x)} + \frac{\pi^2 n_x^2}{6} < \infty$$

for $x < 0$ and $x \neq -\frac{1}{n^2}$ for all $n = 1, 2, 3, \ldots$ This shows that the series converges absolutely on $\mathbb{R} \setminus S$ where

$$S = \{0\} \cup \left\{-1, -\tfrac{1}{4}, -\tfrac{1}{9}, \cdots\right\}.$$

It is clear that the series diverges at each point of S.

(b) Let $\delta > 0$ and suppose $x \notin T_\delta = (-\delta, \delta) \cup \left\{-1, -\frac{1}{4}, -\frac{1}{9}, \cdots\right\}$. Without loss of generality, we may assume $\delta = \frac{1}{K_0^2}$ where K_0 is a positive integer. Let $\varepsilon > 0$ and choose

$$M_0 > \max\left\{ \frac{K_0^2}{\varepsilon} + K_0 ,\ K_0 + 1 \right\}.$$

If $M_0 \leq M < N$ and $x > \delta$ then

$$|S_N(x) - S_M(x)| = \sum_{n=M+1}^{N} \frac{1}{1+n^2 x} < \sum_{n=M+1}^{\infty} \frac{1}{n^2 x} < \frac{1}{\delta} \int_M^{\infty} \frac{1}{x^2}\, dx$$

$$= \frac{1}{M\delta} \leq \frac{K_0^2}{M_0} < \varepsilon .$$

If $M_0 \le M < N$ and $x < -\delta = -\frac{1}{K_0^2} < -\frac{1}{(K_0+1)^2}$ then the argument in part (a) shows that

$$|S_N(x) - S_M(x)| \le \sum_{n=M+1}^{N} \left| \frac{1}{1+n^2(x)} \right| = \sum_{n=M+1}^{N} \frac{1}{-1+n^2(-x)}$$

$$\le \sum_{n=M+1}^{N} \frac{1}{-1+\left(\frac{n}{K_0}\right)^2} = K_0^2 \sum_{n=M+1}^{N} \frac{1}{n^2-K_0^2} < K_0^2 \sum_{n=M+1}^{\infty} \frac{1}{(n-K_0)^2}$$

$$< K_0^2 \int_M^\infty \frac{1}{(x-K_0)^2}\,dx = \frac{K_0^2}{M-K_0} < \varepsilon .$$

By Theorem 7.8, it follows that the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges uniformly on $\mathbb{R} \setminus T_\delta$.

(c) If the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converged uniformly on $(0,b)$ for some $b>0$ then given any $\varepsilon > 0$ there would correspond an integer $N_0 = N_0(\varepsilon) \ge 1$ such that $|f(x) - S_N(x)| = \sum_{n=N+1}^{\infty} \frac{1}{1+n^2x} < \varepsilon$ for all $N \ge N_0$ and all $0 < x < b$. However,

$$\left| f\left(\frac{1}{(N+1)^2}\right) - S_N\left(\frac{1}{(N+1)^2}\right) \right| = \sum_{n=N+1}^{\infty} \frac{1}{1+n^2\left(\frac{1}{(N+1)^2}\right)} > \frac{1}{1+(N+1)^2\left(\frac{1}{(N+1)^2}\right)} = \frac{1}{2}$$

for all $N \ge 1$. Similarly,

$$\left| f\left(\frac{-1}{N^2}\right) - S_N\left(\frac{-1}{N^2}\right) \right| = \sum_{n=N+1}^{\infty} \frac{1}{-1+n^2\left(\frac{1}{N^2}\right)} = \sum_{n=N+1}^{\infty} \frac{N^2}{n^2-N^2} > \frac{N^2}{2N+1} \ge \frac{1}{3}$$

for all $N \ge 1$ so the series does not converge uniformly on $(-b,0)$ for any $b > 0$ either.

(d) Let $\delta > 0$ and observe that each partial sum of $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ is continuous on $\mathbb{R} \setminus T_\delta$ where $T_\delta = (-\delta, \delta) \cup \{-1, -\frac{1}{4}, -\frac{1}{9}, \dots\}$. Since the series converges uniformly on $\mathbb{R} \setminus T_\delta$, Theorem 7.12 shows that f is continuous on $\mathbb{R} \setminus T_\delta$. But $\delta > 0$ is arbitrary so f is continuous on $\mathbb{R} \setminus S$ where $S = \{0\} \cup \{-1, -\frac{1}{4}, -\frac{1}{9}, \cdots\}$. That is, f is continuous on the set of points where the series converges.

(e) f is not bounded on $\mathbb{R} \setminus S$. There are many ways to see this; here is one:

$$f\left(\frac{1}{k^2}\right) = \sum_{n=1}^{\infty} \frac{1}{1+\frac{n^2}{k^2}} = \sum_{n=1}^{\infty} \frac{k^2}{k^2+n^2} \geq \int_1^{\infty} \frac{k^2}{k^2+x^2}\,dx$$

$$= k \operatorname{Arctan}\left(\frac{x}{k}\right)\Big|_1^{\infty} = k\left(\frac{\pi}{2} - \operatorname{Arctan}\left(\frac{1}{k}\right)\right) \longrightarrow \infty$$

as $k \to \infty$.