Mathematics 315 Midterm Exam Spring 2011 Name: Dr. Grow

On this exam, **no proofs are required** to support your answers if you are asked to state a theorem, write a formula, or give an example.

1.(34 pts.) (a) Define the phrase " E is a countable set".

(b) Give an example of a subset E of the real numbers $\mathbb{R}$ which is countable and dense in $\mathbb{R}$.

5 (c) Define the phrase " E is a subset of $\mathbb{R}$ of measure zero".

5 (d) Give an example of a subset E of $\mathbb{R}$ which is countable and not of measure zero or state a theorem showing why this is impossible.

5 (e) Give an example of a subset E of $\mathbb{R}$ which is not countable and has measure zero or state a theorem showing why this is impossible.

5 (f) If f is an increasing real function on $[a,b]$, state a theorem which characterizes the set of points at which f is continuous.

5 (g) If f is an increasing real function on $[a,b]$, state Lebesgue's theorem characterizing the set of points at which f is differentiable.

Solve **ONE** of the following two problems, 2A or 2B. **CIRCLE** the number of the problem that you want me to grade.

2A.(33 pts.) Let f be a bounded real function on $[a,b]$ and let α be an increasing real function on $[a,b]$.

0 (a) Give a **careful and complete** definition of the phrase " f is Riemann-Stieltjes integrable with respect to α on $[a,b]$". (Note: Make sure that symbols such as $U(P,f,\alpha)$, $L(P,f,\alpha)$, $\overline{\int_a^b} f d\alpha$, and $\underline{\int_a^b} f d\alpha$ that appear in your definition are carefully defined.)

7 (b) State a theorem which guarantees the existence of $\int_a^b f d\alpha$.

8 (c) If f is continuous on $[0,1]$ and $\alpha(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}H\left(x-1+\frac{1}{2^n}\right)$, then write a formula for the value of $\int_0^1 f d\alpha$. (Here H denotes the unit Heaviside step function.)

2B.(33 pts.) Let f be a bounded real function on $[a,b]$ and let α be a function of bounded variation on $[a,b]$.

5 (a) Define what it means for α to be of bounded variation on $[a,b]$.

6 (b) Give an example of a function which is differentiable but not of bounded variation on $[a,b]$.

5 (c) State a condition on a differentiable function will guarantee that it is of bounded variation on $[a,b]$.

6 (d) State Jordan's theorem relating functions of bounded variation and increasing functions.

5 (e) How is the Riemann-Stieltjes integral of f with respect to α on $[a,b]$ defined in terms of Riemann-Stieltjes integrals with increasing integrators?

6 (f) If f is Riemann integrable on $[0,1]$ and α is differentiable with α' Riemann integrable on $[0,1]$, then write a formula for the value of $\int^1 f d\alpha$.

#1. (a) E is a countable set if there is a one-to-one function f mapping the positive integers onto the set E.

(b) The rational numbers $\mathbb{Q}$ is an example of a subset of $\mathbb{R}$ which is countable and dense in $\mathbb{R}$.

(c) $E \subseteq \mathbb{R}$ is a set of measure zero if to each $\varepsilon > 0$ there corresponds a countable collection $\{(a_n, b_n)\}_{n=1}^{\infty}$ of open intervals in $\mathbb{R}$ satisfying $E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$ and $\sum_{n=1}^{\infty}(b_n - a_n) < \varepsilon$.

(d) There is no such subset E of $\mathbb{R}$ because if E is countable then E is of measure zero.

(e) The Cantor set P in $[0,1]$ has measure zero and is not countable.

(f) If $f:[a,b] \to \mathbb{R}$ is increasing then the set of discontinuities of f is either finite or countable.

(g) If $f:[a,b] \to \mathbb{R}$ is increasing then the set of points at which f is not differentiable has measure zero.

#2A. (a) Let $P: \quad a = x_0 < x_1 < x_2 < \dots < x_n = b$ be a partition of $[a,b]$ and let $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$ and $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$ for $i = 1, 2, \dots, n$. Form the upper and lower Riemann-Stieltjes sums:

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1}))$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1})).$$

The upper and lower Riemann-Stieltjes integrals are:

$$\overline{\int_a^b} f\,d\alpha = \inf\{U(P, f, \alpha) : P \text{ is a partition of } [a,b]\}$$

$$\underline{\int_a^b} f\,d\alpha = \sup\{L(P, f, \alpha) : P \text{ is a partition of } [a,b]\}.$$

We say that f is Riemann-Stieltjes integrable with respect to α on $[a,b]$ provided

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha .$$

(b) If f is continuous on $[a,b]$ and α is increasing on $[a,b]$ then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

(OR)

If f is monotonic on $[a,b]$ and α is continuous and increasing on $[a,b]$ then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

(c) If f is continuous on $[0,1]$ and $\alpha(x) = \sum_{n=1}^{\infty} 2^{-n} H(x - 1 + 2^{-n})$ then

$$\int_0^1 f\,d\alpha = \sum_{n=1}^{\infty} 2^{-n} f(1 - 2^{-n}).$$

#2B. (a) The function $\alpha : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$ provided there is a real number M such that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le M$$

for all partitions $P: a = x_0 < x_1 < \dots < x_n = b$ of $[a,b]$.

(b) $$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0, \end{cases}$$

is differentiable on $[0,1]$ but f is not of bounded variation on $[0,1]$.

(c) If f is differentiable and $\boxed{f' \text{ is bounded on } [a,b]}$ then f is of bounded variation on $[a,b]$.

(Jordan's Theorem) (d) If α is of bounded variation on $[a,b]$ then there exist increasing real functions α_1 and α_2 on $[a,b]$ such that $\alpha(x) = \alpha_1(x) - \alpha_2(x)$ for all $a \le x \le b$.

(e) Let $\alpha = \alpha_1 - \alpha_2$ where each α_i is increasing on $[a,b]$. If $f \in \mathcal{R}(\alpha_i)$ on $[a,b]$ for $i = 1,2$ then define $\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 - \int_a^b f\,d\alpha_2$.

(f) If $f \in \mathcal{R}[0,1]$ and α is differentiable with $\alpha' \in \mathcal{R}[0,1]$ then $f \in \mathcal{R}(\alpha)$ and $\int_0^1 f\,d\alpha = \int_0^1 f(x)\alpha'(x)\,dx$.

#3A. (a) N is a norm on the real vector space $\mathfrak{X}$ provided N is a real function defined on $\mathfrak{X}$ with the following properties:

(i) $N(\vec{x}) \geq 0$ for all $\vec{x}$ in $\mathfrak{X}$, with equality only if $\vec{x} = 0$;

(ii) $N(c\vec{x}) = |c| N(\vec{x})$ for all $\vec{x}$ in $\mathfrak{X}$ and c in $\mathbb{R}$;

(iii) $N(\vec{x}+\vec{y}) \leq N(\vec{x}) + N(\vec{y})$ for all $\vec{x}$ and $\vec{y}$ in $\mathfrak{X}$.

(b) A sequence $\{\vec{x}_n\}_{n=1}^{\infty}$ in a normed linear space $(\mathfrak{X}, N)$ is convergent provided there exists $\vec{x}$ in $\mathfrak{X}$ such that $\lim_{n\to\infty} N(\vec{x} - \vec{x}_n) = 0$.

(c) A sequence $\{\vec{x}_n\}_{n=1}^{\infty}$ in a normed linear space $(\mathfrak{X}, N)$ is Cauchy provided $N(\vec{x}_n - \vec{x}_m) \to 0$ as m and n tend to infinity.

(d) Convergent sequences are Cauchy sequences in a normed linear space. However, for a general normed linear space, Cauchy sequences need not converge.

(e) A normed linear space $(\mathfrak{X}, N)$ in which every Cauchy sequence is convergent is called a Banach space.

(f) $(C[0,1], \|\cdot\|_1)$, where $\|f\|_1 = \int_0^1 |f|\,dx$ for f in $C[0,1]$, is a normed linear space which is not a Banach space.

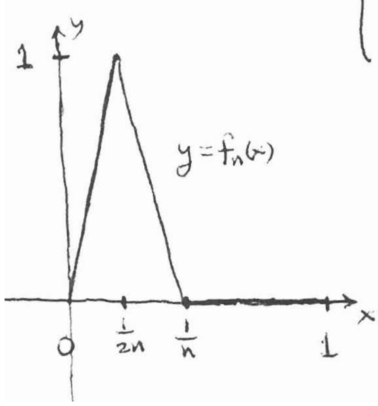
(g) $(C[0,1], \|\cdot\|_u)$, where $\|f\|_u = \sup\{|f(x)| : 0 \leq x \leq 1\}$ for f in $C[0,1]$, is a Banach space.

#3B. (a) $\{f_n\}_{n=1}^{\infty}$ converges to f pointwise on $[a,b]$ provided

$\lim_{n\to\infty} f_n(x) = f(x)$ for each x in $[a,b]$.

(b) $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly on $[a,b]$ provided to each $\varepsilon > 0$ there corresponds an integer $N = N(\varepsilon) \geq 1$ such that $|f_n(x) - f(x)| < \varepsilon$ for all x in $[a,b]$ and all $n \geq N$.

(c) Let $f_n(x) = \begin{cases} 1 - 2n\left|x - \frac{1}{2n}\right| & \text{if } 0 \leq x \leq \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < x \leq 1, \end{cases}$ for $n = 1, 2, 3, \ldots$



and $f(x) = 0$ if $0 \leq x \leq 1$. Then $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on $[0,1]$ but $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to f on $[0,1]$.

(d) Let $f_n : E \to \mathbb{R}$ $(n = 1, 2, 3, \ldots)$ be a sequence of bounded functions on a set E and let $M_n = \sup\{|f_n(x)| : x \in E\}$ $(n = 1, 2, 3, \ldots)$. If $\sum_{n=1}^{\infty} M_n < \infty$ then the sequence of partial sums of $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

(e) Let $\mathcal{A}$ be a family of real functions defined on a set E. $\mathcal{A}$ is called an algebra provided:

(i) if f and g belong to $\mathcal{A}$ then $f+g$ belongs to $\mathcal{A}$;

(ii) if f belongs to $\mathcal{A}$ and c is any real number then cf belongs to $\mathcal{A}$;

(iii) if f and g belong to $\mathcal{A}$ then fg belongs to $\mathcal{A}$.

(f) $\mathcal{A}$ separates points on E if to each pair of dispoints p and q in E there corresponds f in $\mathcal{A}$ such that $f(p) \neq f(q)$.

(g) $\mathcal{A}$ vanishes at no point of E if to each point p in E there corresponds f in $\mathcal{A}$ such that $f(p) \neq 0$.

(h) Let $\mathcal{A}$ be an algebra of real continuous functions on a compact metric space K. If $\mathcal{A}$ separates points on K and if $\mathcal{A}$ vanishes at no point of K then to each continuous function $f : K \to \mathbb{R}$ there corresponds a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $\mathcal{A}$ such that $f_n \to f$ uniformly on K.

Math 315

Midterm Exam

Spring 2011

n: 14

mean: 77.1

standard deviation: 15.2

Distribution of Scores:

Range	Letter Grade	Frequency
80 - 100	A	7
60 - 79	B	6
40 - 59	C	1
0 - 39	F	0