Mathematics 325 Exam I Summer 2012 Name: Dr. Grow

1.(25 pts) Classify the type (elliptic, parabolic, hyperbolic) of

$$u_{xx} - u_{xy} + 3u_{yy} - 3u_{yx} = \sin(x+y)$$

and find the general solution, if possible, in the xy-plane.

The PDE is equivalent to $u_{xx} - 4u_{xy} + 3u_{yy} = \sin(x+y)$ so $B^2 - 4AC =$

3 pts. $(-4)^2 - 4(1)(3) = 4 > 0$. Therefore the PDE is $\boxed{\text{hyperbolic}}$. The PDE can be rewritten in factored operator form as

3pts. $$(*) \qquad \left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = \sin(x+y) \qquad \text{so} \qquad \alpha = 1, \beta = -3, \gamma = 1, \delta = -1.$$

The suggested change of coordinates is

2 pts. $$(**) \qquad \begin{cases} \xi = -(\beta x - \alpha y) = -(-3x - y) = 3x + y \\ \eta = -(\delta x - \gamma y) = -(-x - y) = x + y. \end{cases}$$

Using the chain rule and $(**)$ we see that, as operators,

2pts. $$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

Substituting in $(*)$ gives

$$\left[\left(\cancel{3\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right) - 3\left(\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right)\right]\left[3\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}} - \left(\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}}\right)\right]u = \sin(\eta)$$

3pts. $$\left(-2\frac{\partial}{\partial \eta}\right)\left(2\frac{\partial}{\partial \xi}\right)u = \sin(\eta)$$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = -\frac{1}{4}\sin(\eta).$$

Integrating with respect to η holding ξ fixed gives

3 pts. $$\frac{\partial u}{\partial \xi} = \int \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right)d\eta = \int -\frac{1}{4}\sin(\eta)\,d\eta = \frac{1}{4}\cos(\eta) + c_1(\xi).$$

Integrating with respect to ξ holding η fixed yields

(OVER)

3pts. $$u = \int \frac{\partial u}{\partial \xi}\, d\xi = \int \left[\frac{1}{4}\cos(\eta) + c_1(\xi) \right] d\xi = \frac{\xi}{4}\cos(\eta) + \int c_1(\xi)\, d\xi + c_2(\eta).$$

That is,

$$u = \frac{\xi}{4}\cos(\eta) + f(\xi) + g(\eta)$$

where $\boxed{f \text{ and } g \text{ are } C^2\text{-functions of a single real variable.}}$ Substituting from (**), we see that, as a function of x and y,

3pts. $$\boxed{u(x,y) = \frac{1}{4}(3x+y)\cos(x+y) + f(3x+y) + g(x+y)}.$$

2.(25 pts.) (a) Find the general solution of $(1+x)u_x + u_y = 0$.

(b) What is the largest region in the xy – plane in which the solution to the equation in (a) is valid?

(c) Sketch some of the characteristic curves of the equation in (a).

(a) Any solution $u = u(x,y)$ to the PDE $a(x,y)u_x + b(x,y)u_y = 0$ is constant along its characteristic curves given by $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$. The characteristic curves of the PDE in (a) are thus

6 pts.

$$\frac{dy}{dx} = \frac{1}{1+x}.$$

Separating variables gives

$$dy = \frac{dx}{1+x}$$

and integrating both sides yields

5 pts.

$$y = \int dy = \int \frac{dx}{1+x} = \ln|1+x| + c.$$

Along such a curve $y = y(x)$, any solution u to $(1+x)u_x + u_y = 0$ is constant, so

$$u(x, y(x)) = u(x, \ln|1+x| + c) = u(0, \ln(1) + c) = u(0, c) = f(c).$$

10 pts.

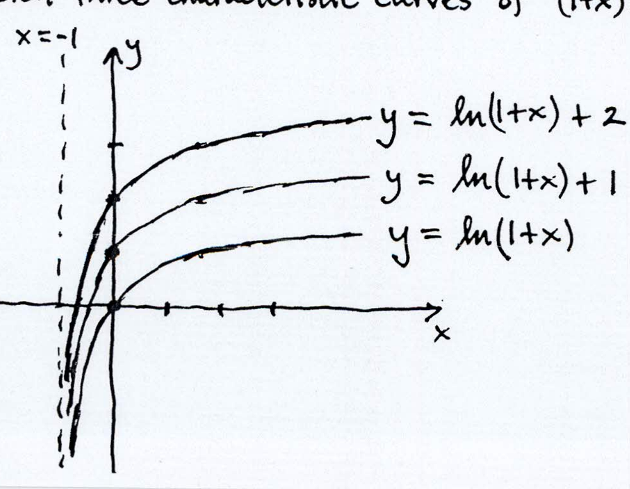
Thus, the general solution of $(1+x)u_x + u_y = 0$ is $\boxed{u(x,y) = f(y - \ln|1+x|)}$ where f is a C^1-function of a single real variable.

(b) The solution in (a) is valid if $1 + x \neq 0$; i.e. $x \neq -1$. Therefore the largest region (i.e. connected open set) in the xy-plane where the solution is valid is either the half-plane $\boxed{x > -1}$ or the half-plane $\boxed{x < -1.}$

2 pts.

(c) We sketch three characteristic curves of $(1+x)u_x + u_y = 0$ in the region $x > -1$.

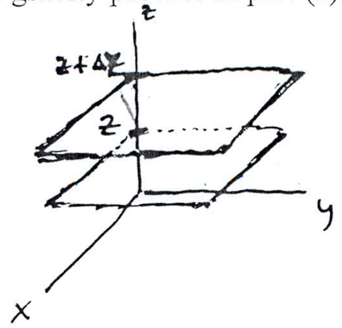
2 pts.



3.(25 pts.) (a) (Diffusion-in-the-presence-of-gravity) In the absence of diffusion, some particles suspended in a liquid medium would travel downward at the constant rate $V > 0$ due to gravity. Taking into account diffusion, derive the equation of motion governing the concentration of the particles. (Hints: Assume homogeneity in the horizontal directions x and y, and let the $z-$axis point upward. Recall that diffusing particles move from regions of higher concentration to regions of lower concentration, and the rate of motion is proportional to the gradient of the concentration.)

(b) Find the boundary condition on an impermeable plane at $z = a$ for the diffusion-in-the-presence-of-gravity process in part (a).

(18 total pts.)



(a) Denote the concentration of particles at height z and time t by $u(z,t)$. Then the total mass of the particles between horizontal planes at height $z+\Delta z$ and height z at time t is

3pts. $$M(t) = \int_z^{z+\Delta z} u(\xi,t)\,d\xi$$ so the time rate of change of the

mass is $$\frac{dM}{dt} = \frac{d}{dt}\int_z^{z+\Delta z} u(\xi,t)\,d\xi = \int_z^{z+\Delta z} \frac{\partial u}{\partial t}(\xi,t)\,d\xi.$$ Alternatively the 3pts.

time rate of change of the mass M is given by

$\frac{dM}{dt}$ = influx of mass at height $z+\Delta z$ − outflux of mass at height z

$$= \left[k u_z(z+\Delta z,t) + V u(z+\Delta z,t) \right] - \left[k u_z(z,t) + V u(z,t) \right]$$ 6pts.

mass diffusion rate at height $z+\Delta z$ is proportional to gradient of concentration there

rate of migration of mass caused by gravity at height $z+\Delta z$

mass diffusion rate at height z is proportional to gradient of concentration there

rate of migration of mass caused by gravity at height z

Equate the two expressions for $\frac{dM}{dt}$, rearrange, and divide through by Δz to obtain

3pts. $$\frac{1}{\Delta z}\int_z^{z+\Delta z} u_t(\xi,t)\,d\xi = k\left[\frac{u_z(z+\Delta z,t) - u_z(z,t)}{\Delta z}\right] + V\left[\frac{u(z+\Delta z,t) - u(z,t)}{\Delta z}\right].$$

Letting Δz approach 0 in the above identity produces

(OVER)

$$u_t(z,t) = ku_{zz}(z,t) + Vu_z(z,t)$$

or equivalently

3pts $$\boxed{u_t - Vu_z - ku_{zz} = 0}$$

(7 total pts.)

(b) Suppose there is an impermeable membrane at height $z=a$. Arguing as in part (a), the mass between the planes at heights $a+\Delta z$ and a at time t is given by

1 pt. $$M(t) = \int_a^{a+\Delta z} u(\xi,t)\,d\xi \,,$$

2 pts. so $$\frac{dM}{dt} = \int_a^{a+\Delta z} u_t(\xi,t)\,d\xi .$$ But also

$$\frac{dM}{dt} = \text{influx of mass at } a+\Delta z \; - \text{ outflux of mass at } a$$

2 pts. $$= ku_z(a+\Delta z,t) + Vu(a+\Delta z,t) \quad - \; 0 .$$

Equating the two expressions for $\frac{dM}{dt}$ leads to

1 pt. $$\int_a^{a+\Delta z} u_t(\xi,t)\,d\xi = ku_z(a+\Delta z,t) + Vu(a+\Delta z,t) .$$

Letting Δz approach 0 in the above identity produces the boundary condition at $z=a$:

1 pt. $$\boxed{0 = ku_z(a,t) + Vu(a,t)} .$$

4.(25 pts.) Consider the undamped vibrations of an infinite string modeled by

$$\rho u_{tt} - T u_{xx} = 0 \quad \text{for} \quad -\infty < x < \infty,\ 0 < t < \infty,$$
$$u(x,0) = \varphi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for} \quad -\infty < x < \infty,$$

where the mass density ρ and the tension T are positive constants and the initial position φ and the initial velocity ψ are C^2 and C^1 functions, respectively, of a single real variable.

(a) If $\rho = T$, $\varphi(x) = e^{-x^2}$, and $\psi(x) = 2xe^{-x^2}$, use d'Alembert's formula to show that the motion of the string consists solely of a wave traveling to the right along the $x-$axis. Sketch profiles of the motion at the instants $t = 0,\ 1,$ and 2.

(b) For general ρ, T, φ, and ψ, what nontrivial relation will guarantee that the motion of the string consists solely of a wave traveling to the right along the $x-$axis? Show that your relation accomplishes this.

(17 total pts.)

(a) $\rho u_{tt} - T u_{xx} = 0$ is equivalent to $u_{tt} - c^2 u_{xx} = 0$ where $c = \sqrt{\frac{T}{\rho}}$.

1 pt. Therefore, if $\rho = T$ then $c = 1$. The solution to

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{if } -\infty < x < \infty,\ 0 < t < \infty, \\ u(x,0) = e^{-x^2} \text{ and } u_t(x,0) = 2xe^{-x^2} & \text{if } -\infty < x < \infty, \end{cases}$$

is given by d'Alembert's formula:

2 pts. $$u(x,t) = \frac{1}{2}\left[\varphi(x-ct) + \varphi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\,d\xi$$

2 pts. $$= \frac{1}{2}\left[e^{-(x-t)^2} + e^{-(x+t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t} 2\xi e^{-\xi^2}\,d\xi .$$

If we make the substitution $w = -\xi^2$ in the integral then $dw = -2\xi d\xi$ so

$\int 2\xi e^{-\xi^2} d\xi = \int e^{w}(-dw) = -e^{w} = -e^{-\xi^2}$. Thus

7 pts. $$u(x,t) = \frac{1}{2}\left[e^{-(x-t)^2} + e^{-(x+t)^2}\right] - \frac{1}{2}e^{-\xi^2}\Big|_{\xi = x-t}^{x+t}$$

2 pts. $$= \frac{1}{2}\left[e^{-(x-t)^2} + \cancel{e^{-(x+t)^2}}\right] - \frac{1}{2}\left[\cancel{e^{-(x+t)^2}} - e^{-(x-t)^2}\right]$$

1 pt. $$\boxed{u(x,t) = e^{-(x-t)^2}}.$$

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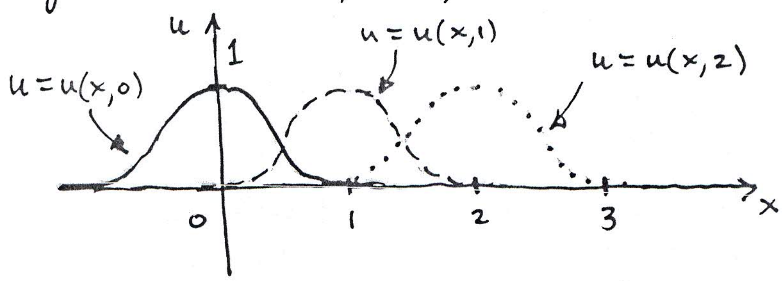
The profiles of the displacement function $u(x,t) = e^{-(x-t)^2}$ of the string at times $t=0$, $t=1$, and $t=2$ are shown below.

2 pts.



Clearly the wave travels to the right down the x-axis with unit speed.

(8 total pts.)

(b) For general ρ, T, φ, and ψ, the solution to

$$\begin{cases} \rho u_{tt} - T u_{xx} = 0 & \text{if } -\infty < x < \infty,\ 0 < t < \infty, \\ u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x) & \text{if } -\infty < x < \infty, \end{cases}$$

is given by d'Alembert's formula with $c = \sqrt{\frac{T}{\rho}}$. This can written in the form

1 pt.

$$u(x,t) = \frac{1}{2}\left[\varphi(x-ct) + \varphi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\,d\xi$$

2 pts.

$$= \underbrace{\frac{1}{2}\varphi(x-ct) + \frac{1}{2c}\int_{x-ct}^{0} \psi(\xi)\,d\xi}_{\text{a wave traveling to the right}} + \underbrace{\frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_{0}^{x+ct} \psi(\xi)\,d\xi}_{\text{a wave traveling to the left}}.$$

The solution is a wave traveling to the right along the x-axis if and only if the portion of the solution traveling to the left is constant; i.e.

1 pt.

$$\frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(\xi)\,d\xi = \text{constant} \qquad \text{for all } -\infty < x < \infty \text{ and } 0 < t < \infty$$

But this condition is equivalent to

1 pt.

$$\frac{1}{2}\varphi(z) + \frac{1}{2c}\int_0^{z} \psi(\xi)\,d\xi = \text{constant} \qquad \text{for all real } z.$$

Differentiating this identity with respect to z leads to

2 pts.

$$\frac{1}{2}\varphi'(z) + \frac{1}{2c}\psi(z) = 0 \qquad \text{for all real } z,$$

or equivalently

1 pt.

$$\boxed{\varphi'(z) = -\sqrt{\frac{\rho}{T}}\,\psi(z) \qquad \text{for all real } z}.$$

Math 325

Exam I

Summer 2012

$n = 16$

$\mu = 55.7$ (mean)

$\sigma = 16.7$ (standard deviation)

Distribution of Scores

Range	Graduate Grade	Undergraduate Grade	Frequency
87-100	A	A	1
73-86	B	B	2
60-72	C	B	3
50-59	C	C	5
0-49	F	D	5