Mathematics 325 | Final Exam Fall 2010 | Name: Dr. Grow

The last two pages of this exam consist of a table of Fourier transforms and some convergence theorems for Fourier series. Furthermore, you may find the following formulas useful on this exam.

$$\int_0^x \frac{dz}{a+b\cos(z)} = \frac{2}{\sqrt{a^2-b^2}} \operatorname{Arctan}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{x}{2}\right)\right) \text{ if } a^2 > b^2 \text{ and } -\pi < x < \pi$$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\varphi)}\frac{\partial}{\partial \varphi}\left(\sin(\varphi)\frac{\partial u}{\partial \varphi}\right) + \frac{1}{r^2\sin^2(\varphi)}\frac{\partial^2 u}{\partial \theta^2}$$

1.(33 pts.) Classify the partial differential equation $u_{xx} + 2u_{yy} - 3u_{xy} + u_x + 2u_y = 0$ as hyperbolic, elliptic, parabolic, or none of these. If it is possible, find the general solution of this partial differential equation in the xy – plane. If it is not possible, please explain why this is so.

2.(34 pts.) (a) Use Fourier transform methods to derive the formula

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+t-\tau} f(s,\tau)\,ds\,d\tau$$

for a solution to the inhomogeneous wave equation in the xt – plane,

$$u_{tt} - u_{xx} = f(x,t),$$

satisfying homogeneous initial conditions: $u(x,0) = 0 = u_t(x,0)$ for $-\infty < x < \infty$

(b) Compute a solution to $u_{tt} - u_{xx} = xt$ in the xt – plane which satisfies homogeneous initial conditions.

(c) Is the solution to the problem in part (b) unique? Justify your answer.

3.(33 pts.) Solve $u_t - u_{xx} = 0$ in the upper half-plane $-\infty < x < \infty$, $0 < t < \infty$, subject to the initial condition $u(x,0) = e^{-x^2}$ for $-\infty < x < \infty$.

4.(34 pts.) The material in a thin circular disk of unit radius has reached a steady-state temperature distribution. The material is held at 100 degrees Centigrade on the top half of the disk's edge and at 0 degrees Centigrade on the bottom half of its edge.

(a) Write the partial differential equation and boundary condition(s) governing the steady-state temperature function of the material.

(b) Write a formula for the steady-state temperature function of the material.

(c) What is the steady-state temperature of the material at the center of the disk? Justify your answer.

(d) What is the steady-state temperature of the material at a general point in the disk? (You must evaluate any integral expressions for credit on this part of the problem.)

5.(33 pts.) (a) Show that the Fourier sine series $\sum_{n=1}^{\infty} b_n \sin(n\pi\theta)$ of $f(\theta) = 4\theta(1-\theta)$ on $[0,1]$ is

$$4\theta(1-\theta) \approx \frac{32}{\pi^3}\sum_{k=1}^{\infty}\frac{\sin((2k-1)\pi\theta)}{(2k-1)^3}$$

(b) Show that the Fourier sine series of f converges uniformly to f on the interval $[0,1]$.

(c) Use the results of the previous parts of this problem to find the sum of the series $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^6}$

6.(33 pts.) Solve $\nabla^2 u = 0$ in the unit cube $0 < x < 1,\ 0 < y < 1,\ 0 < z < 1,$ given that

$$u(x, y, 1) = 16xy(1-x)(1-y) \quad \text{for} \quad 0 \le x \le 1,\ 0 \le y \le 1,$$

and that u satisfies homogeneous Dirichlet boundary conditions on the other five faces of the cube. (You may find the results of problem 5 useful.)

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$
A.	$\begin{cases}1 & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases}1 & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2+a^2}\quad(a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases}x & \text{if } 0<x\le b,\\ 2b-x & \text{if } b<x<2b,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases}e^{-ax} & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases}e^{ax} & \text{if } b<x<c,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases}e^{iax} & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases}e^{iax} & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2}\quad(a>0)$	$\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x}\quad(a>0)$	$\begin{cases}0 & \text{if } \lvert\xi\rvert\ge a,\\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert<a.\end{cases}$

Convergence Theorems

Consider the eigenvalue problem

(1) $X''(x)+\lambda X(x)=0$ in $a<x<b$ with any symmetric boundary conditions

and let $\Phi=\{X_1,X_2,X_3,...\}$ be the complete orthogonal set of eigenfunctions for (1). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$f(x)\approx\sum_{n=1}^{\infty}A_nX_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle}\qquad (n=1,2,3,...).$$

Theorem 2. (Uniform Convergence) If
(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and
(ii) f satisfies the given symmetric boundary conditions,
then the Fourier series of f converges uniformly to f on $[a,b]$

Theorem 3. (L^2 – Convergence) If

$$\int_a^b |f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)
(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.
(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty}A_nX_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x

#1. $\quad u_{xx} - 3u_{xy} + 2u_{yy} + u_x + 2u_y = 0 \qquad (*)$

$B^2 - 4AC = (-3)^2 - 4(1)(2) = 1 > 0$. Therefore the PDE $(*)$ is $\boxed{\text{hyperbolic.}}$ In order to solve the PDE we factor the differential operator associated with the second order terms in $(*)$:

5 pts. to here.

$$\frac{\partial^2}{\partial x^2} - 3\frac{\partial^2}{\partial x \partial y} + 2\frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right).$$

Therefore $(*)$ is equivalent to

$$\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u + \left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right)u = 0.$$

to here

The suggested substitution in this case is

$$\xi = 2x + y$$
$$\eta = x + y.$$

6 pts to here

Then the chain rule implies

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

2 pts here.

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

so

$$\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - 2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = -\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = \frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + 2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = 4\frac{\partial}{\partial \xi} + 3\frac{\partial}{\partial \eta}.$$

pts. ere.

Consequently $(*)$ is equivalent to $\displaystyle -\frac{\partial^2 u}{\partial \xi \partial \eta} + 4\frac{\partial u}{\partial \xi} + 3\frac{\partial u}{\partial \eta} = 0. \qquad (**)$

Unfortunately, the techniques that allow us to solve $u_{\xi\eta} + \alpha u_{\xi} = 0$ or $u_{\xi\eta} + \beta u_{\eta} =$ do not apply to (**). Therefore we conclude that it is not possible to solve (*), at least with the techniques of solution covered in this course.

Note 1: Using the method of separation of variables in (**) leads to formal solutions of the form

$$u(\xi,\eta) = \sum_{n=1}^{\infty} c_n e^{\left(\frac{3\lambda_n}{4+\lambda_n}\right)\xi - \lambda_n \eta}$$

where $c_1, c_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ are arbitrary constants.

Note 2: Using the change of dependent variable $u(x,y) = v(x,y)e^{10x+7y}$ in (*) produces the equivalent equation

$$(\dagger) \qquad \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)v + 12v = 0.$$

Then the change of independent variables

$$\xi = x$$

$$\eta = 3x + 2y$$

in $(\dagger)$ leads to the equivalent equation

$$(\dagger\dagger) \qquad \frac{\partial^2 v}{\partial \xi^2} - \frac{\partial^2 v}{\partial \eta^2} + 12v = 0.$$

But $(\dagger\dagger)$ is a special case of the telegraph equation

$$\frac{\partial^2 v}{\partial \xi^2} - c^2\frac{\partial^2 v}{\partial \eta^2} + 2\beta\frac{\partial v}{\partial \xi} + \alpha v = 0$$

with $c = 1$, $\beta = 0$, and $\alpha = 12$. (See p. 335f in "PDEs with BVPs with Applications" (third edition) by Mark Pinsky.) It can be shown that the solution to (††) satisfying the initial conditions

$$(\dagger\dagger\dagger) \qquad v(\eta, 0) = f_1(\eta) \text{ and } \frac{\partial v}{\partial \xi}(\eta, 0) = f_2(\eta) \qquad \text{for } -\infty < \eta < \infty$$

is

$$v(\eta, \xi) = \frac{1}{2}\int_{-\xi}^{\xi} f_2(\eta + s) J_0\left(\sqrt{12\xi^2 - 12s^2}\right) ds$$

$$+ \frac{1}{2}\left(f_1(\eta + \xi) + f_1(\eta - \xi)\right)$$

$$- \frac{\sqrt{12}}{2}\int_{-\xi}^{\xi} f_2(\eta + s) J_1\left(\sqrt{12\xi^2 - 12s^2}\right) ds$$

where J_0 and J_1 are the Bessel functions of orders 0 and 1, respectively. (See Pinsky's book cited above, pp. 461–462 and exercise 8.5.4.) Of course, this is far beyond what was covered in our Math 325 class but I thought people might want to know how to solve this PDE.

#2 (a) See problem 3 of Exam II.

(b) See the bonus portion of problem 3 on Exam II where it is shown that $\boxed{u(x,t) = \frac{xt^3}{6}.}$ 7 pts.

(c) Suppose that u is a solution to the problem in part (a) such that, for each fixed t in $(-\infty,\infty)$, u, u_t, u_x, u_{xt}, u_{xx}, and u_{tt} are square-integrable functions of x and $\lim\limits_{|x|\to\infty} u_x(x,t) = 0 = \lim\limits_{|x|\to\infty} u_t(x,t)$. Then u is unique.

<u>Reason:</u> Let $v = v(x,t)$ be another such solution to the problem in part (a) and define $w(x,t) = u(x,t) - v(x,t)$ for all real x and t. Then w solves $w_{tt} - w_{xx} = 0$ in the xt-plane and $w(x,0) = 0 = w_t(x,0)$ for $-\infty < x < \infty$.
Consider the energy function of w:

$$E(t) = \int_{-\infty}^{\infty} \left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t) \right] dx \qquad \text{if} \quad -\infty < t < \infty .$$

[Note that the square-integrability hypotheses imply that the energy function of w is finite for all t in $(-\infty,\infty)$.] The square-integrability hypotheses allow differentiation of the energy function of w:

$$\frac{dE}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t) \right] dx = \int_{-\infty}^{\infty} \left[w_t(x,t) w_{tt}(x,t) + w_x(x,t) w_{xt}(x,t) \right] dx$$

and the decay conditions and integration by parts show

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} w_t(x,t) w_{tt}(x,t)\,dx + w_t(x,t) w_x(x,t) \Big|_{t=-\infty}^{\infty} - \int_{-\infty}^{\infty} w_t(x,t) w_{xx}(x,t)\,dx$$

$$= \int_{-\infty}^{\infty} w_t(x,t) \left[w_{tt}(x,t) - w_{xx}(x,t) \right] dx \qquad = 0,$$

so $E = E(t)$ is a constant function on $(-\infty,\infty)$. Since

$$E(0) = \int_{-\infty}^{\infty} \left[\tfrac{1}{2} w_t^2(x,0) + \tfrac{1}{2} w_x^2(x,0) \right] dx = 0$$

it follows that $E(t) = 0$ for all $-\infty < t < \infty$. By the vanishing theorem, $w_t(x,t) = 0 = w_x(x,t)$ for all $-\infty < x < \infty$ and each fixed t in $(-\infty, \infty)$. Then $w(x,t) =$ constant in the xt-plane and $w(x,0) = 0$ implies $w(x,t) = 0$ for all $-\infty < x < \infty$ and all $-\infty < t < \infty$. That is, $v \equiv u$ so the solution to the problem in part (a) satisfying the square-integrability and decay conditions is unique.

$\boxed{\#3}$ See problem 1 on Exam III where it is shown that

$$\boxed{u(x,t) = \frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}}\ .$$

$\boxed{\#4}$ (a) Let $u = u(r;\theta)$ denote the steady-state temperature of the material in the disk at the point whose polar coordinates are $(r;\theta)$. Then u satisfies

6 pts

$$\boxed{\begin{aligned} &\nabla^2 u = 0 \quad \text{in} \quad 0 \le r < 1,\ -\pi \le \theta \le \pi, \\ &u(1;\theta) = h(\theta) = \begin{cases} 100 & \text{if} \quad 0 < \theta < \pi, \\ 0 & \text{if} \quad -\pi < \theta < 0. \end{cases} \end{aligned}}$$

(b) Poisson's formula

$$u(r;\theta) = \frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{h(\varphi)\,d\varphi}{a^2 - 2ar\cos(\theta-\varphi)+r^2} \qquad (0 \le r < a, \ -\pi \le \theta \le \pi$$

applied to the problem in part (a) gives

$$\boxed{u(r;\theta) = \frac{1-r^2}{2\pi}\int_{0}^{\pi} \frac{100\,d\varphi}{1 - 2r\cos(\theta-\varphi)+r^2}} \qquad (0 \le r < 1, \ -\pi \le \theta \le \pi).$$

(c) We substitute $r=0$ in the formula in part (b) (or apply the mean value theorem for harmonic functions) to obtain the steady-state temperature of the material at the center of the disk:

$$u(0;\theta) = \frac{1}{2\pi}\int_0^{\pi} 100\,d\varphi = \boxed{50}.$$

(d) Let $z = \varphi - \theta$ in the formula for u in part (b). Then

$$(*) \quad u(r;\theta) = \frac{1-r^2}{2\pi}\int_{-\theta}^{\pi-\theta} \frac{100\,dz}{1+r^2-2r\cos(z)}$$

$$= \frac{50(1-r^2)}{\pi}\left(\int_0^{\pi-\theta} \frac{dz}{1+r^2-2r\cos(z)} - \int_0^{-\theta} \frac{dz}{1+r^2-2r\cos(z)}\right).$$

Suppose that $0 < \theta < \pi$. Then $0 < \pi - \theta < \pi$ and $-\pi < -\theta < 0$ so by the integration formula on the first page of the exam with $a = 1+r^2$ and $b = -2r$, we have

$$u(r;\theta) = \frac{50(1-r^2)}{\pi}\cdot\frac{2}{\sqrt{(1+r^2)^2-(2r)^2}}\left[\operatorname{Arctan}\left(\sqrt{\frac{1+r^2+2r}{1+r^2-2r}}\tan\left(\frac{\pi-\theta}{2}\right)\right) - \operatorname{Arctan}\left(\sqrt{\frac{1+r^2+2r}{1+r^2-2r}}\tan\left(-\frac{\theta}{2}\right)\right)\right]$$

Simplifying yields

$$(**) \qquad u(r;\theta) = \frac{100}{\pi}\left[\operatorname{Arctan}\left(\frac{1+r}{1-r}\cot(\theta/2)\right) + \operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(\theta/2)\right)\right] \quad \text{if} \quad 0<\theta<\pi.$$

Suppose $-\pi<\theta<0$. Then $0<-\theta<\pi$ and $\pi<\pi-\theta<2\pi$ so

$$(***) \qquad -\int_0^{-\theta} \frac{dz}{1+r^2-2r\cos(z)} = \frac{2}{1-r^2}\operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(\theta/2)\right)$$

as before, and

Let $w = z-\pi$

$$(****) \qquad \int_0^{\pi-\theta} \frac{dz}{1+r^2-2r\cos(z)} = \int_0^{\pi} \frac{dz}{1+r^2-2r\cos(z)} + \int_{\pi}^{\pi-\theta} \frac{dz}{1+r^2-2r\cos(z)}$$

$$= \frac{2}{1-r^2}\cdot\frac{\pi}{2} + \int_0^{-\theta}\frac{dw}{1+r^2+2r\cos(w)}$$

$$= \frac{2}{1-r^2}\cdot\frac{\pi}{2} + \frac{2}{1-r^2}\cdot\operatorname{Arctan}\left(\frac{1-r}{1+r}\tan(-\theta/2)\right)$$

$$= \frac{2}{1-r^2}\cdot\frac{\pi}{2} - \frac{2}{1-r^2}\operatorname{Arctan}\left(\frac{1-r}{1+r}\tan(\theta/2)\right).$$

Using $(*)$, $(***)$, and $(****)$ we have

$$(*****) \qquad u(r;\theta) = \frac{100}{\pi}\left[\frac{\pi}{2} - \operatorname{Arctan}\left(\frac{1-r}{1+r}\tan(\theta/2)\right) + \operatorname{Arctan}\left(\frac{}{1-r}\tan(\theta/2)\right)\right]$$

o here.

provided $-\pi<\theta<0$. Collecting $(**)$ and $(*****)$ produces the solution to the steady-state temperature problem in part (a):

$$\boxed{u(r;\theta) = \begin{cases} \dfrac{100}{\pi}\left[\operatorname{Arctan}\left(\dfrac{1+r}{1-r}\cot(\theta/2)\right) + \operatorname{Arctan}\left(\dfrac{1+r}{1-r}\tan(\theta/2)\right)\right] & \text{if } 0<\theta<\pi \\[2ex] \dfrac{100}{\pi}\left[\dfrac{\pi}{2} - \operatorname{Arctan}\left(\dfrac{1-r}{1+r}\tan(\theta/2)\right) + \operatorname{Arctan}\left(\dfrac{1+r}{1-r}\tan(\theta/2)\right)\right] & \text{if } -\pi<\theta<0 \end{cases}}$$

<u>Note</u>: Using the identity $\tan(\alpha+\beta) = \dfrac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$, one can show that the solution to the problem in part (a) of problem 4 is

$$\boxed{u(r,\theta) = 50 + \frac{100}{\pi}\operatorname{Arctan}\left(\frac{2r\sin\theta}{1-r^2}\right)}\,.$$

$\boxed{\#5}$ (a) See part (a) of problem 1 on Exam III.

(b) See part (c) of problem 1 on Exam III.

(c) See part (e) of problem 1 on Exam III where it was shown that

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{(2k-1)^6} = \frac{\pi^6}{960}}\,.$$

$\boxed{\#6}$ We seek the solution of the boundary value problem :

① $\quad u_{xx} + u_{yy} + u_{zz} = 0$ if $0<x<1,\ 0<y<1,\ 0<z<1,$

②-③ $\quad u(x,0,z) = 0 = u(x,1,z)$ if $0\le x\le 1,\ 0\le z\le 1,$

④-⑤ $\quad u(0,y,z) = 0 = u(1,y,z)$ if $0\le y\le 1,\ 0\le z\le 1,$

⑥-⑦ $\quad u(x,y,0) = 0$ and $u(x,y,1) = 16xy(1-x)(1-y)$ if $0\le x\le 1, 0\le y\le 1$

We use the method of separation of variables. We seek nontrivial solutions to the homogeneous part of the problem, ①-②-③-④-⑤-⑥, of the form $u(x,y,z) = X(x)Y(y)Z(z)$. Substituting this expression for u in ① and simplifying yields $\dfrac{Z''(z)}{Z(z)} + \dfrac{Y''(y)}{Y(y)} = -\dfrac{X''(x)}{X(x)} = \text{constant} = \lambda$ and then $\dfrac{Z''(z)}{Z(z)} - \lambda = -\dfrac{Y''(y)}{Y(y)} = \text{constant} = \mu$. Substituting $u(x,y,z) = X(x)Y(y)Z(z)$ in ④-⑤ and using the fact that u is not identically zero produces $X(0) = 0 = X(1)$. Similarly ②-③ leads to $Y(0) = 0 = Y(1)$ and ⑥ leads to $Z(0) = 0$. Therefore we are led to the system

$$\begin{cases} X''(x) + \lambda X(x) = 0 \text{ if } 0 < x < 1, \; X(0) = 0 = X(1), \\ Y''(y) + \mu Y(y) = 0 \text{ if } 0 < y < 1, \; Y(0) = 0 = Y(1), \\ Z''(z) - (\lambda + \mu) Z(z) = 0 \text{ if } 0 < z < 1, \; Z(0) = 0. \end{cases}$$

The solutions to X and Y eigenvalue problems with Dirichlet boundary conditions are, respectively,

$$\lambda_\ell = (\ell\pi)^2, \; X_\ell(x) = \sin(\ell\pi x) \quad (\ell = 1,2,3,\dots)$$

$$\mu_m = (m\pi)^2, \; Y_m(y) = \sin(m\pi y) \quad (m = 1,2,3,\dots).$$

The solution, up to a constant multiple, of the corresponding Z-problem when $\lambda = \lambda_\ell$ and $\mu = \mu_m$ is $Z_{\ell,m}(z) = \sinh\left(\pi z\sqrt{\ell^2 + m^2}\right)$ $(\ell = 1,2,3,\dots;\ m = 1,2,3,\dots)$

By superposition, a formal solution to ①-②-③-④-⑤-⑥ is

=1 pts to here

$$u(x,y,z) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} c_{\ell,m} \sin(\ell\pi x)\sin(m\pi y)\sinh\left(\pi z\sqrt{\ell^2 + m^2}\right)$$

where the $c_{\ell,m}$ are arbitrary constants. To satisfy ⑦ we must have

$$4x(1-x)4y(1-y) = u(x,y,1) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} b_{l,m} \sin(l\pi x)\sin(m\pi y)$$

for all $0 \le x \le 1$ and $0 \le y \le 1$; here $b_{l,m} = c_{l,m} \sinh\left(\pi\sqrt{l^2+m^2}\right)$ for $l = 1,2,3,\ldots$ and $m = 1,2,3,\ldots$. But the $b_{l,m}$ must be the Fourier coefficients of the function $g(x,y) = 4x(1-x)4y(1-y)$ on the unit square $0 \le x \le 1$, $0 \le y \le 1$, with respect to the orthogonal set

$$\varphi_{l,m}(x,y) = \sin(l\pi x)\sin(m\pi y) \qquad (l = 1,2,3,\ldots\,;\ m = 1,2,3,\ldots)$$

on the unit square. That is,

$$b_{l,m} = \frac{\langle g, \varphi_{l,m}\rangle}{\langle \varphi_{l,m}, \varphi_{l,m}\rangle} = \frac{\int_0^1\int_0^1 4x(1-x)4y(1-y)\sin(l\pi x)\sin(m\pi y)\,dx\,dy}{\int_0^1\int_0^1 \sin^2(l\pi x)\sin^2(m\pi y)\,dx\,dy}$$

$$= \left(\frac{\int_0^1 4x(1-x)\sin(l\pi x)\,dx}{\int_0^1 \sin^2(l\pi x)\,dx}\right)\cdot\left(\frac{\int_0^1 4y(1-y)\sin(m\pi y)\,dy}{\int_0^1 \sin^2(m\pi y)\,dy}\right)$$

$$= \left(2\int_0^1 4x(1-x)\sin(l\pi x)\,dx\right)\cdot\left(2\int_0^1 4y(1-y)\sin(m\pi y)\,dy\right).$$

By problem 5 we have

$$2\int_0^1 4x(1-x)\sin(l\pi x)\,dx = \begin{cases} 0 & \text{if } l = 2j \text{ is even,} \\ \dfrac{32}{\pi^3(2j-1)^3} & \text{if } l = 2j-1 \text{ is odd,} \end{cases}$$

$$2\int_0^1 4y(1-y)\sin(m\pi y)\,dy = \begin{cases} 0 & \text{if } m = 2k \text{ is even,} \\ \dfrac{32}{\pi^3(2k-1)^3} & \text{if } m = 2k-1 \text{ is odd.} \end{cases}$$

Therefore

$$b_{2j-1,\,2k-1} = \frac{(32)}{\pi^6 (2j-1)^3 (2k-1)^3} \quad (j=1,2,3,\dots \text{ and } k=1,2,3,\dots)$$ and all

other $b_{\ell,m}$ are zero. Consequently

$$c_{2j-1,\,2k-1} = \frac{(32)^2}{\pi^6 (2j-1)^3 (2k-1)^3 \sinh\left(\pi\sqrt{(2j-1)^2+(2k-1)^2}\right)} \quad (j=1,2,3,\dots \text{ and } k=1,2,3,\dots)$$

and all other $c_{\ell,m}$ are zero. Thus a solution to 1-2-3-4-5-6-7 is

$$\boxed{u(x,y,z) = \frac{1024}{\pi^6} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\sin((2j-1)\pi x)\sin((2k-1)\pi y)\sinh\left(\pi z\sqrt{(2j-1)^2+(2k-1)^2}\right)}{(2j-1)^3 (2k-1)^3 \sinh\left(\pi\sqrt{(2j-1)^2+(2k-1)^2}\right)}}$$

Math 325

Fall 2010

Final Exam

number taking exam: 36

mean: 131.0

median: 140

standard deviation: 37.1

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
174 – 200	A	A	6
146 – 173	B	B	8
120 – 145	C	B	10
100 – 119	C	C	3
0 – 99	F	D	9