Mathematics 325 Exam I Fall 2012 Name: Dr. Grow

1.(33 pts.) (a) Find the general solution of $\left(1+x^2\right)u_x + xyu_y = 0$ (1) in the $xy-$plane.

(b) Identify by name the characteristic curves of the partial differential equation in part (a) and sketch several of them.

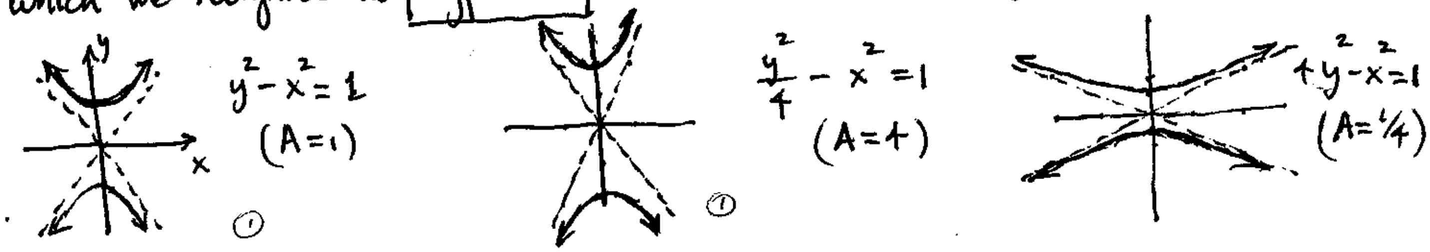
(c) Find the solution of the partial differential equation in part (a) satisfying the auxiliary condition $u(0,y) = y^6$ (2) for all real y.

22 pts. (a) The characteristic equation of (1) is $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} = \frac{xy}{1+x^2}$ ②. Separating ③ variables gives $\frac{dy}{y} \overset{②}{=} \frac{x\,dx}{1+x^2}$ and integrating both sides yields ② $\ln|y| = \frac{1}{2}\ln(1+x^2)+c_1$ ①. Multiplying through by 2, rearranging and exponentiating both sides produces $y^2 = A(1+x^2)$ ④ where A is a positive constant. At each point along such a characteristic curve, a solution to (1) must be constant. Hence, along a characteristic curve,

$$u(x,y) = u(0,y(0)) = u(0, \pm\sqrt{A}) = f(A). \quad ④$$

But then at an arbitrary point (x,y), $\boxed{u(x,y) = f\left(\frac{y^2}{1+x^2}\right)}$ ④ where f is a C^1-function of a single real variable.

4 pts. (b) The characteristic curves of (1) are $\boxed{y^2 = A(1+x^2)}$ or equivalently $\frac{y^2}{A} - x^2 = 1$ which we recognize as $\boxed{\text{hyperbolas}}$ ② with center $(0,0)$ and asymptotes $y = \pm\sqrt{A}x$.



7 pts. (c) In order to satisfy the auxiliary equation (2), we must choose the function f in the general solution so that $y^6 = u(0,y) = f\left(\frac{y^2}{1+0^2}\right) \overset{②}{=} f(y^2)$ for all real y. In other words $f(z) \overset{③}{=} z^3$ for all real $z \geq 0$. This means that

$$\boxed{u(x,y) =} f\left(\frac{y^2}{1+x^2}\right) = \boxed{\left(\frac{y^2}{1+x^2}\right)^3} \quad ③$$

for all (x,y) in the plane.

2.(34 pts.) Classify the following second order partial differential equations as linear or nonlinear. In the case of the nonlinear equations, circle a term that if deleted would produce a linear equation. In the case of the linear equations, classify each as elliptic, parabolic, or hyperbolic and find the general solution whenever possible. (2)

(a) $u_{xx} - 3u_{xy} + u_y u_y = 0$ (1) $\boxed{\text{nonlinear}}$

(b) $u_{xx} - 4u_{xy} + 5u_{yy} - 3u_y = 0$ $\quad B^2 - 4AC = 16 - 20 < 0$ $\boxed{\text{(1) linear, elliptic}}$ (2)

(c) $u_{xx} + 3uu_{yy} + u_{xy} = 0$ (1) $\boxed{\text{nonlinear}}$ (2)

(d) $u_{xx} + u_{yy} - 2u_{xy} + u_x - u_y = 0$ $\quad B^2 - 4AC = 4 - 4 = 0$ $\boxed{\text{linear, (1) parabolic}}$ (2)

22 pts. We find the general solution of (d) by first rewriting the equation as

$$\left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right)u + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = 0 \quad \text{or} \quad \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 u + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = 0. \quad (2)$$

This suggests the change-of-coordinates:

$$\xi = -(\beta x - \alpha y) = -(-x-y) = x+y \quad (2)$$

$$\eta = \alpha x + \beta y = x - y = x - y. \quad (2)$$

The chain rule gives $\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \overset{(1)}{=} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} \overset{(1)}{=} \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$. Hence $\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) = 2\frac{\partial}{\partial \eta}$ (1), so the p.d.e. in (d) can be written as $4\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial u}{\partial \eta} \overset{(2)}{=} 0$. Writing $v = \frac{\partial u}{\partial \eta}$ this becomes $4\frac{\partial v}{\partial \eta} + 2v = 0 \iff \frac{\partial v}{\partial \eta} + \frac{1}{2}v \overset{(2)}{=} 0$. An integrating factor is $\mu = e^{\int p(\eta)d\eta} = e^{\int \frac{1}{2}d\eta} \overset{(2)}{=} e^{\eta/2 + \cancel{c}}$. Then $e^{\frac{\eta}{2}}\frac{\partial v}{\partial \eta} + \frac{1}{2}e^{\frac{\eta}{2}}v = 0$ or $\frac{\partial}{\partial \eta}\left(e^{\frac{\eta}{2}}v\right) \overset{(2)}{=} 0$

so $e^{\frac{\eta}{2}}v \overset{(1)}{=} c_1(\xi)$. But then $\frac{\partial u}{\partial \eta} = v = c_1(\xi)e^{-\eta/2}$ so integrating with respect to η holding ξ fixed gives $u = \int c_1(\xi)e^{-\eta/2}d\eta = -2c_1(\xi)e^{-\eta/2} + c_2(\xi)$. In other words $u \overset{(2)}{=} f(\xi)e^{-\eta/2} + g(\xi)$ where f and g are any C^2-functions of a single real variable. Thus

$$\boxed{u(x,y) \overset{(2)}{=} f(x+y)e^{\frac{1}{2}(y-x)} + g(x+y)}$$

is the general solution of (d).

3.(33 pts.) A homogeneous solid material occupying the hollowed out cylinder

$$C=\{(x,y,z)\in\mathbb{R}^3 : 1\le x^2+y^2\le 4,\ 0\le z\le 2\}$$

is completely insulated and its initial temperature at position (x,y,z) in C is $50/\sqrt{x^2+y^2}$.

(a) Write, without proof or derivation, the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) in C and time $t\ge 0$.

(b) Use the divergence theorem to help show that the heat energy

$$H(t)=\iiint_C c\rho u(x,y,z,t)\,dV$$

of the material in C at time t is a constant function of time. Here c and ρ denote the specific heat and mass density, respectively, of the material in C.

(c) Compute the steady-state temperature that the material in C reaches after a long time.

15 pts. (a)

$$\begin{cases} (1)\quad u_t - k\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right)=0 & \text{if } (x,y,z) \text{ is in the interior of } C \text{ and } t>0,\\ (2)\quad (\nabla u\cdot\vec{n}=)\ \dfrac{\partial u}{\partial n}=0 & \text{if } (x,y,z) \text{ belongs to the boundary of } C \text{ and } t\ge 0,\\ (3)\quad u(x,y,z,0)=\dfrac{50}{\sqrt{x^2+y^2}} & \text{if } (x,y,z) \text{ is in } C. \end{cases}$$

(The first equation says u obeys the heat equation, the second expresses the fact that the body is completely insulated, and the third gives the initial temperature distribution in the body.)

9 pts. (b)

$$\frac{dH}{dt}=\frac{d}{dt}\iiint_C c\rho u(x,y,z,t)\,dV=\iiint_C c\rho\frac{\partial u}{\partial t}(x,y,z,t)\,dV \overset{(1)}{=} \iiint_C c\rho k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)dV$$

$$= c\rho k\iiint_C \nabla\cdot(\nabla u)\,dV \overset{\text{Divergence Theorem}}{=} c\rho k\iint_{\partial C}\nabla u\cdot\vec{n}\,dS \overset{(2)}{=} c\rho k\iint_{\partial C} 0\,dS = 0.$$

Therefore the heat energy of the material in C is a constant function of t.

9 pts. (c) Let $U=\lim_{t\to\infty} u(x,y,z,t)$ be the steady-state temperature of the material in C. Since $H(t)$ is constant,

$$H(0)=\lim_{t\to\infty} H(t)=\iiint_C c\rho \lim_{t\to\infty} u(x,y,z,t)\,dV=\iiint_C c\rho U\,dV = c\rho U \operatorname{vol}(C) = c\rho U 6\pi.$$

But

$$H(0)=\iiint_C c\rho u(x,y,z,0)\,dV = \int_0^2\int_0^{2\pi}\int_1^2 c\rho\,\frac{50}{r}\,r\,dr\,d\theta\,dz = c\rho 50\cdot 4\pi.$$

Consequently,

$$U=\frac{c\rho 50\cdot 4\pi}{c\rho 6\pi}=\boxed{\frac{100}{3}}.$$

Math 325

Exam I

Fall 2012

mean: 60.9

median: 65

standard deviation: 22.4

number taking exam: 36

Distribution of Scores

Range	Graduate Grade	Undergraduate Grade	Frequency
87 – 100	A	A	4
73 – 86	B	B	7
60 – 72	C	B	9
50 – 59	C	C	4
0 – 49	F	D	12