Mathematics 325 | Exam I Summer 2013 | Name: Dr. Grow

1.(30 pts.) Solve the partial differential equation $u_x - 2xy^2 u_y = 0$ subject to the auxiliary condition $u(0,y) = y^2$ if $y > 0$.

Characteristic curves of $a(x,y)u_x + b(x,y)u_y = 0$ obey $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$. In the p.d.e. above, this is

4 pts. to here.

$$\frac{dy}{dx} = \frac{-2xy^2}{1}.$$

Separating variables and integrating, we find

14 pts. to here.

$$\frac{1}{y} = \int -\frac{1}{y^2}\,dy = \int 2x\,dx = x^2 + c$$

or equivalently $y = \frac{1}{x^2+c}$. Along such a curve $y = y(x)$, a solution u to the p.d.e. is constant. Hence, for all real x,

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$$u(x, y(x)) = u\left(x, \frac{1}{x^2+c}\right) = u\left(0, \frac{1}{c}\right) = f(c).$$

But the curves $\frac{1}{y} - x^2 = c$ "pave" the upper half-plane $y > 0$; i.e. there is precisely one such curve passing through each point in the upper half-plane. Consequently the general solution to $u_x - 2xy^2 u_y = 0$ in the upper half-plane is

22 pts. to here.

$$u(x,y) = f(c) = f\left(\frac{1}{y} - x^2\right)$$

where f is a continuously differentiable function of a single real variable. To satisfy the auxiliary condition we must have $y^2 = u(0,y) = f\left(\frac{1}{y} - 0^2\right) = f\left(\frac{1}{y}\right)$

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for all $y > 0$. Letting $w = \frac{1}{y}$ this becomes $f(w) = \frac{1}{w^2}$ for all $w > 0$.

Hence

30 pts. to here

$$u(x,y) = f\left(\frac{1}{y} - x^2\right) = \boxed{\frac{1}{\left(\frac{1}{y} - x^2\right)^2}}$$

is the solution to the problem. An equivalent form of the solution is

$$u(x,y) = \frac{1}{\left(\frac{1}{y} - x^2\right)^2} = \frac{1}{\left(\frac{1 - x^2 y}{y}\right)^2} = \boxed{\frac{y^2}{(1 - x^2 y)^2}}.$$

2.(40 pts.) (a) Classify the type - elliptic, hyperbolic, or parabolic - of the p.d.e. $u_{tt} - c^2 u_{xx} = 0$.
(b) Derive the general solution of $u_{tt} - c^2 u_{xx} = 0$ in the xt-plane.
(c) Use the result of (a) to help derive a formula for the solution to the initial value problem

$$u_{tt} - c^2 u_{xx} = 0 \text{ if } -\infty < x < \infty,\ 0 \le t < \infty,$$
$$u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x) \text{ if } -\infty < x < \infty.$$

Here φ is an arbitrary twice continuously differentiable function of a single real variable and ψ is an arbitrary continuously differentiable function of a single real variable.
(d) Apply the result of (b) to solve the initial value problem

$$u_{tt} - u_{xx} = 0 \text{ if } -\infty < x < \infty,\ 0 \le t < \infty,$$
$$u(x,0) = x^2 \text{ and } u_t(x,0) = 4xe^{-x^2} \text{ if } -\infty < x < \infty.$$

4 (a) $B^2 - 4AC = 0^2 - 4(-c^2)(1) = 4c^2 > 0$. The equation is $\boxed{\text{hyperbolic}}$.

(b) We write the p.d.e. in differential operator form: $\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0$.
Factoring the operator gives $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$. This suggests the change-of-coordinates

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$$\begin{cases} \xi = -(\beta t - \alpha x) = -(-ct - x) = ct + x, \\ \eta = \delta t - \gamma x = ct - x. \end{cases}$$

Consequently

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$$\begin{cases} \frac{\partial}{\partial x} = \overset{1}{\cancel{\frac{\partial \xi}{\partial x}}}\frac{\partial}{\partial \xi} + \overset{-1}{\cancel{\frac{\partial \eta}{\partial x}}}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial t} = \overset{c}{\cancel{\frac{\partial \xi}{\partial t}}}\frac{\partial}{\partial \xi} + \overset{c}{\cancel{\frac{\partial \eta}{\partial t}}}\frac{\partial}{\partial \eta} = c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}. \end{cases}$$

Substituting these expressions into the factored-operator form of the p.d.e. gives

$$\left[\cancel{c\frac{\partial}{\partial \xi}} + c\frac{\partial}{\partial \eta} - c\left(\cancel{\frac{\partial}{\partial \xi}} - \frac{\partial}{\partial \eta}\right)\right]\left[c\frac{\partial}{\partial \xi} + \cancel{c\frac{\partial}{\partial \eta}} + c\left(\frac{\partial}{\partial \xi} - \cancel{\frac{\partial}{\partial \eta}}\right)\right]u = 0$$

$$\left(2c\frac{\partial}{\partial \eta}\right)\left(2c\frac{\partial}{\partial \xi}\right)u = 0$$

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$$\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0.$$

Integrating with respect to η holding ξ fixed gives $\frac{\partial u}{\partial \xi} = \int 0\,d\eta = c_1(\xi)$.
Integrating again, this time with respect to ξ holding η fixed, yields
$u = \int c_1(\xi)d\xi = c_3(\xi) + c_2(\eta)$. As a function of x and t,

19 $\boxed{u(x,t) = f(x+ct) + g(-x+ct)}$ where f and g are twice continuously differentiable

functions of a single real variable.

(c) The initial conditions imply

(i) $\varphi(x) = u(x,0) = f(x) + g(-x)$ if $-\infty < x < \infty$,

(ii) $\psi(x) = u_t(x,0) = cf'(x) + cg'(-x)$ if $-\infty < x < \infty$.

Differentiating (i) yields

(i′) $\varphi'(x) = f'(x) - g'(-x)$

23 (ii) $\psi(x) = cf'(x) + cg'(-x)$.

Multiplying (i′) by c and adding to (ii) gives

$$c\varphi'(x) + \psi(x) = 2cf'(x) \quad \text{if } -\infty < x < \infty.$$

Dividing by $2c$ and integrating over $[0,x]$ gives

27 (✱) $$\frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds + k = f(x) \quad \text{if } -\infty < x < \infty$$

where k is an arbitrary constant. Substituting in (i) gives

$$g(-x) = \varphi(x) - f(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds - k,$$

or equivalently

31 (✱✱) $$g(x) = \frac{1}{2}\varphi(-x) - \frac{1}{2c}\int_0^{-x} \psi(s)\,ds - k \quad \text{if } -\infty < x < \infty.$$

Consequently, substituting from (✱) and (✱✱) gives

$$u(x,t) = f(x+ct) + g(-x+ct) = \frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(s)\,ds + \frac{1}{2}\varphi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi(s)\,ds$$

or equivalently,

35 $$\boxed{u(x,t) = \frac{1}{2}\Big[\varphi(x+ct) + \varphi(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds}$$ (d'Alembert's formula).

(d) Applying d'Alembert's formula with $c=1$, $\varphi(x) = x^2$, and $\psi(x) = 4xe^{-x^2}$, the solution of the I.V.P. is

38 $$u(x,t) = \frac{1}{2}\Big[(x+t)^2 + (x-t)^2\Big] + \frac{1}{2}\int_{x-t}^{x+t} 4se^{-s^2}\,ds = \frac{1}{2}\Big[(x+t)^2 + (x-t)^2\Big] - e^{-s^2}\Big|_{x-t}^{x+t}$$

40 $$\boxed{u(x,t) = \frac{1}{2}\Big[(x+t)^2 + (x-t)^2\Big] + e^{-(x-t)^2} - e^{-(x+t)^2} = x^2 + t^2 + 2e^{-(x^2+t^2)}\sinh(2xt)}.$$

3.(30 pts.) Consider the Neumann boundary value problem

$$\text{(1)}\quad \nabla^2 u = f(x,y,z) \quad \text{in } D: x^2+y^2+z^2<1,$$

$$\text{(2)}\quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D: x^2+y^2+z^2=1.$$

(a) Are solutions to this problem unique? Justify your answer.
(b) Discuss the physical interpretation for the result of (a) in the case of heat flow.
(c) Use the divergence theorem to show that

$$\iiint_D f(x,y,z)\,dxdydz = 0$$

is a necessary condition if the Neumann problem is to have a solution.
(d) Discuss the physical interpretation for the result of (c) in the case of heat flow.

3 pts. (a) No, solutions to the B.V.P. are not unique because if $u=u(x,y,z)$ solves (1)-(2) then clearly $u=u(x,y,z)+c$ solves (1)-(2) for any constant c.

6 pts. (b) In the case of heat flow, $u(x,y,z)$ denotes the steady-state temperature at position (x,y,z) in $\bar{D}=D\cup\partial D$. The result of part (a) means that there is no such thing as the "absolute" temperature at position (x,y,z) in $\bar{D}$. According to this model, the only physically meaningful quantity in this closed system (see (d)) is the temperature difference measured from an arbitrarily chosen standard like the freezing point of water. (Of course, the model (1)-(2) for steady-state temperatures was derived using the assumptions of continuum mechanics. Quantum mechanical effects leading to an absolute temperature are not incorporated in this model.)

12 pts. (c) Suppose that $u=u(x,y,z)$ is a solution to (1)-(2). Then

$$0=\iint_{\partial D} 0\,dS \overset{\text{by (2)}}{=} \iint_{\partial D}\frac{\partial u}{\partial n}\,dS \overset{\text{definition of } \frac{\partial u}{\partial n}}{=} \iint_{\partial D}\nabla u\cdot\vec{n}\,dS \overset{\text{Divergence Theorem}}{=} \iiint_D \nabla\cdot(\nabla u)\,dV \overset{\text{definitions}}{=} \iiint_D \nabla^2 u\,dV \overset{\text{by (1)}}{=} \iiint_D f(x,y,z)\,dV.$$

Therefore a necessary condition for the existence of a solution to (1)-(2) is $\iiint_D f(x,y,z)\,dV=0$.

9 pts. (d) We view (1)-(2) as modeling the steady-state temperature distribution $u=u(x,y,z)$ at position (x,y,z) in $\bar{D}=D\cup\partial D$. The presence of f in (1) indicates sources $(f(x,y,z)>0)$ and/or sinks $(f(x,y,z)<0)$ of heat energy within the material occupying $\bar{D}$. The condition (2) corresponds to an insulated boundary, ∂D, of D. That is, there is no

flux of heat energy across ∂D so the system $\overline{D}$ is "isolated" or "closed". The result of part (b) means that in order for solutions to exist to ①-②, the average value

$$\frac{1}{vol(D)} \iiint_D f(x,y,z)\,dV$$

of the source/sink term f must be zero. This is just another way of saying that the total heat energy of the closed system in $\overline{D}$ must be conserved.

Math 325

Exam I

Summer 2013

number of exams : 8

mean : 83.8

median :

standard deviation: 8.5

Distribution of Scores

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87 – 100	A	A	5
73 – 86	B	B	2
60 – 72	C	B	1
50 – 59	C	C	0
0 – 49	F	D	0