Mathematics 325 Exam I Name: Dr. Grow

Summer 2014

1.[35] (a) Show that $u(x,y)=(x+y)^3$ is a solution of $yu_x - xu_y = 3(y-x)(x+y)^2$ in the xy-plane.

(b) Find the general solution of $yu_x - xu_y = 3(y-x)(x+y)^2$ in the xy-plane.

5 pts. (a) $yu_x - xu_y = y\left(3(x+y)^2\right) - x\left(3(x+y)^2\right) \overset{\checkmark}{=} 3(y-x)(x+y)^2$

(b) The general solution to the nonhomogeneous PDE is of the form $u = u_0 + u_p$ where $u_0 = u_0(x,y)$ is the general solution of the associated homogeneous equation $yu_x - xu_y = 0$ and $u_p = u_p(x,y) = (x+y)^3$ is a particular solution to the nonhomogeneous PDE. The characteristic curves for $a(x,y)u_x + b(x,y)u_y = 0$ are given by $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$. Therefore

5 pts.

11 pts. to here. $\frac{dy}{dx} = \frac{-x}{y}$ describes the characteristic curves of $yu_x - xu_y = 0$. Separating variables and integrating gives

$$\frac{y^2}{2} = \int y\,dy = \int -x\,dx = -\frac{x^2}{2} + c_1 \Rightarrow x^2 + y^2 = c \quad (c = 2c_1).$$

17 pts. to here.

Along such a characteristic curve, $u_0(x, y(x)) = u(0, y(0)) = u(0, \pm\sqrt{c}) = f(c)$.

23 pts. to here. Therefore $u_0(x,y) = f(x^2+y^2)$ where $f \in C^1(\mathbb{R})$. Thus

$$\boxed{u(x,y) = f(x^2+y^2) + (x+y)^3}$$

27 pts. to here.

is the general solution of $yu_x - xu_y = 3(y-x)(x+y)^2$ where f is an arbitrary continuously differentiable function of a single real variable.

2.[35] Classify the following differential equations as elliptic, parabolic, hyperbolic, or nonlinear and find the general solution in the xy – plane whenever possible.

5 (a) $u_{xx} - u_{yy} + uu_x = 0$ Nonlinear

4 (b) $u_{xx} + 4u_{yy} - 4u_{xy} + 25u = 0$ Parabolic since $B^2 - 4AC = (-4)^2 - 4(1)(4) = 0$.

4 (c) $u_{xx} + u_{xy} + 3u_{yy} + u_{yx} = 0$ Elliptic since $B^2 - 4AC = (2)^2 - 4(1)(3) = -8 < 0$.

To solve (b) we rewrite it as $\left(\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x \partial y} + 4\frac{\partial^2}{\partial y^2}\right)u + 25u = 0$, or equivalently,

4 pts. to here. $\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)u + 25u = 0$. This suggests the change-of-coordinates

8 pts. to here. $\begin{cases} \xi = x - 2y, \\ \eta = 2x + y. \end{cases}$ Then the chain rule implies $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial v}{\partial \xi} + 2\frac{\partial v}{\partial \eta}$

and $\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} = -2\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$. That is, as operators we have

12 pts. to here. $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta} - 2\frac{\partial}{\partial \xi}$. Substituting in (b) gives

$$\left[\left(\frac{\partial}{\partial \xi} + \cancel{2\frac{\partial}{\partial \eta}}\right) - 2\left(\cancel{\frac{\partial}{\partial \eta}} - 2\frac{\partial}{\partial \xi}\right)\right]\left[\left(\frac{\partial}{\partial \xi} + \cancel{2\frac{\partial}{\partial \eta}}\right) - 2\left(\cancel{\frac{\partial}{\partial \eta}} - 2\frac{\partial}{\partial \xi}\right)\right]u + 25u = 0$$

$$\left(5\frac{\partial}{\partial \xi}\right)\left(5\frac{\partial}{\partial \xi}\right)u + 25u = 0$$

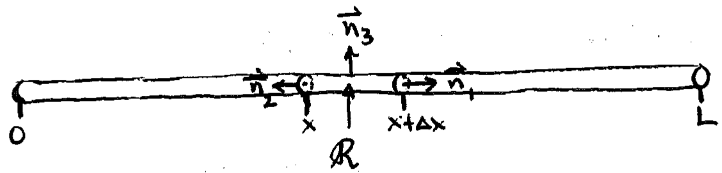
16 pts. to here.
$$\frac{\partial^2 u}{\partial \xi^2} + u = 0.$$

20 pts. to here. The general solution of this PDE is $u = c_1(\eta)\cos(\xi) + c_2(\eta)\sin(\xi)$. Therefore

24 pts. to here.
$$\boxed{u(x,y) = f(2x+y)\cos(x-2y) + g(2x+y)\sin(x-2y)}$$

is the general solution of (b) where f and g are arbitrary, twice-continuously differentiable functions of a single real variable.

3.[30] In the interior of a very thin rod, heat exchange takes place according to Fourier's law - the flow is from hot to cold regions and the velocity of the flow is proportional to the temperature gradient. On the sides of the thin rod, heat exchange takes place according to Newton's law of cooling - the heat flux is proportional to the temperature difference. Assume the rod is surrounded by a medium of constant temperature T_0. Derive the partial differential equation satisfied by the temperature $u(x,t)$ at position x and time t in the rod.



Consider the material occupying the region R between positions x and $x+\Delta x$. We consider the rod sufficiently thin that we may safely neglect the variation of the temperature u over a cross-section of the rod. Denote by H the heat energy contained in the material occupying R. Then

$$(1) \qquad -\frac{dH}{dt} = \text{flux of heat through the boundary of } R$$

4 pts. to here.

$$= \iint_{\partial R} \vec{V}\cdot\vec{n}\,d\sigma = \iint_{\text{Right End}} -Ku_x(x+\Delta x,t)\,d\sigma + \iint_{\text{Left End}} Ku_x(x,t)\,d\sigma + \iint_{\text{Lateral Wall}} \mu(u(\xi,t)-T_0)\,d\sigma$$

7 pts. to here. where $\vec{V}$ = the velocity of the heat flux $= \begin{cases} -K\nabla u & \text{for conductive heat loss on ends,} \\ \mu(u-T_0)\vec{n}_3 & \text{for convective heat loss on sides,} \end{cases}$

and K and μ are real constants. Performing the integrals in the right member of (1) gives

$$\text{10 pts. to here } (2) \qquad -\frac{dH}{dt} = -KAu_x(x+\Delta x,t) + KAu_x(x,t) + \mu P\int_x^{x+\Delta x}(u(\xi,t)-T_0)\,d\xi.$$

Here A is the cross-sectional area and P is the perimeter of the cross-section. On the other hand,

$$\text{13 pts. to here. } (3) \qquad H(t) = \iiint_R E(\vec{x},t,u(\vec{x},t))\,d\vec{x}$$

where $E = E(\vec{x},t,u)$ is the heat energy density function for the material occupying R. Differentiating (3) gives

$$\text{16 pts. to here. } (4) \qquad \frac{dH}{dt} = \iiint_R \frac{\partial}{\partial t}\left(E(\vec{x},t,u(\vec{x},t))\right)dV = \iiint_R \frac{\partial E}{\partial u}\cdot\frac{\partial u}{\partial t}\,dV.$$

(OVER)

0 pts. to here. For normal temperature ranges and typical materials, $\frac{\partial E}{\partial u} \cong$ constant $= c\rho$ where c is the specific heat and ρ is the density of the material occupying $\mathcal{R}$. Substituting in (4) and integrating in the y and z coordinates leads to

3 pts. to here (5)
$$\frac{dH}{dt} = \iiint_{\mathcal{R}} c\rho u_t \, dV = c\rho A \int_x^{x+\Delta x} u_t(\xi, t)\, d\xi$$

where A is the cross-sectional area. Comparing (3) and (5) produces

6 pts. to here. (6)
$$c\rho A \int_x^{x+\Delta x} u_t(\xi,t)\,d\xi = KA u_x(x+\Delta x,t) - KA u_x(x,t) - \mu P \int_x^{x+\Delta x} [u(\xi,t) - T_0]\,d\xi .$$

Dividing through (6) by Δx and letting $\Delta x \to 0$ yields

$$c\rho A \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} u_t(\xi,t)\,d\xi = KA \lim_{\Delta x \to 0} \frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} - \mu P \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} [u(\xi,t) - T_0]\,d\xi$$

or

$$c\rho A u_t(x,t) = KA u_{xx}(x,t) - \mu P (u(x,t) - T_0)) .$$

Equivalently

30 pts. to here.
$$\boxed{u_t(x,t) = \frac{K}{c\rho} u_{xx}(x,t) - \frac{\mu P}{c\rho A}(u(x,t) - T_0)}$$

where K is the conductivity, μ is the surface conductance, c is the specific heat, ρ is the density, A is the cross-sectional area, and P is the perimeter of a cross-section.

Math 325

Exam I

Summer 2014

$n = 17$

$\mu = 58.8$

$\sigma = 22.8$

Range	Grad. Letter Grad	Undergrad. Letter Grade	Frequency
87 – 100	A	A	2
73 – 86	B	B	5
60 – 72	C	B	1
50 – 59	C	C	0
0 – 49	F	D	9