Mathematics 325 | Exam II Summer 2013 | Name: Dr. Grow

1.(20 pts.) Solve the eigenvalue problem

$$X''(x)+\lambda X(x)\overset{①}{=}0 \text{ if } 0<x<1, \text{ subject to } X(0)\overset{②}{=}0\overset{③}{=}X'(1).$$

You may assume that λ is a real number, but you must examine in complete detail all the cases for λ.

(10 pts.) Case $\lambda>0$ (say $\lambda=\beta^2$ where $\beta>0$): Then ① is $X''(x)+\beta^2 X(x)=0$ with general solution $X(x)=c_1\cos(\beta x)+c_2\sin(\beta x)$. In this case $X'(x)=-\beta c_1\sin(\beta x)+\beta c_2\cos(\beta x)$ so ② implies $0=X(0)=c_1$ and ③ implies $0=X'(1)=-\beta \cancel{c_1}^{0}\sin(\beta)+\beta c_2\cos(\beta)=\beta c_2\cos(\beta)$. In order for a nontrivial solution to exist we must have $\cos(\beta)=0$ so $\beta=\left(\frac{2n+1}{2}\right)\pi$ for some integer $n\geq 0$. Hence the positive eigenvalues are $\lambda_n=\beta_n^2=\left(\frac{2n+1}{2}\right)^2\pi^2$ and the corresponding eigenfunctions are $X_n(x)=\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)$ where $n=0,1,2,\ldots$

(4 pts.) Case $\lambda=0$: Then ① is $X''(x)=0$ with general solution $X(x)=c_1x+c_2$. In this case $X'(x)=c_1$ so ② implies $0=X(0)=c_2$ and ③ implies $0=X'(1)=c_1$. Therefore there are no nontrivial solutions in this case; i.e. $\lambda=0$ is not an eigenvalue.

(6 pts.) Case $\lambda<0$ (say $\lambda=-\alpha^2$ where $\alpha>0$): Then ① is $X''(x)-\alpha^2X(x)=0$ with general solution $X(x)=c_1\cosh(\alpha x)+c_2\sinh(\alpha x)$. In this case, $X'(x)=\alpha c_1\sinh(\alpha x)+\alpha c_2\cosh(\alpha x)$ so ② implies $0=X(0)=c_1$ and ③ implies $0=X'(1)=\alpha c_1\sinh(\alpha)+\alpha c_2\cosh(\alpha)=\alpha c_2\cosh(\alpha)$. But $\alpha\cosh(\alpha)>\alpha>0$ so $c_2=0$. Consequently there are only trivial solutions in this case; i.e. there are no negative eigenvalues.

2.(40 pts.) Find a solution to the boundary problem

$$u_{tt}-u_{xx}\overset{(1)}{=}0 \quad \text{if } 0<x<1,\ 0<t<\infty,$$
$$u(0,t)\overset{(2)}{=}0\overset{(3)}{=}u_x(1,t) \quad \text{if } 0\le t<\infty,$$
$$u(x,0)\overset{(5)}{=}\frac{1}{2}\sin^3\left(\frac{\pi x}{2}\right),\ u_t(x,0)\overset{(4)}{=}0 \ \text{ if } 0\le x\le 1.$$

Notes: You may assume that the solution to the eigenvalue problem in # 1 of this exam is

$$\lambda_n=\left(\left(\frac{2n+1}{2}\right)\pi\right)^2,\ X_n(x)=\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right) \quad (n=0,1,2,\ldots).$$

You may also find useful the identity $\sin^3(\theta)=(3\sin(\theta)-\sin(3\theta))/4$.

Bonus (10 pts.): Show that there is at most one solution to the boundary problem above.

We use the method of separation of variables. We seek nontrivial solutions of the homogeneous portion of the problem (1)-(2)-(3)-(4) of the form $u(x,t)=X(x)T(t)$. Substituting in (1) gives $X(x)T''(t)-X''(x)T(t)=0$ and rearranging yields

$$-\frac{X''(x)}{X(x)}=-\frac{T''(t)}{T(t)}=\text{constant}=\lambda.$$

Substituting in (2) produces $X(0)T(t)=0$ for all $t\ge 0$. But $u(x,t)=X(x)T(t)$ is not the zero function so it follows that $X(0)=0$. Similar arguments involving (3) and (4) imply $X'(1)=0$ and $T'(0)=0$. Collecting this data gives the coupled system

(15 pts.)

$$\begin{cases} X''(x)+\lambda X(x)\overset{(6)}{=}0, & X(0)\overset{(7)}{=}0\overset{(8)}{=}X'(1),\\ T''(t)+\lambda T(t)\overset{(9)}{=}0, & T'(0)\overset{(10)}{=}0.\end{cases}$$

According to problem 1 of this exam, the eigenvalue problem (6)-(7)-(8) has solutions $\lambda_n=\left(\frac{2n+1}{2}\right)^2\pi^2$ and $X_n(x)=\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)$ for $n=0,1,2,\ldots$

Substituting $\lambda=\lambda_n$ in (9) yields $T_n''(t)+\left(\frac{2n+1}{2}\right)^2\pi^2T_n(t)=0$ and the general solution is $T_n(t)=c_1\cos\left(\left(\frac{2n+1}{2}\right)\pi t\right)+c_2\sin\left(\left(\frac{2n+1}{2}\right)\pi t\right)$. Consequently $T_n'(t)=$ (4 pts.)

$-\left(\frac{2n+1}{2}\right)\pi c_1\sin\left(\left(\frac{2n+1}{2}\right)\pi t\right)+\left(\frac{2n+1}{2}\right)\pi c_2\cos\left(\left(\frac{2n+1}{2}\right)\pi t\right)$ and (10) implies $0=T_n'(0)=\left(\frac{2n+1}{2}\right)\pi c_2$

(3) so $c_2=0$. Hence $T_n(t)=\cos\left(\left(\frac{2n+1}{2}\right)\pi t\right)$, up to a constant factor. Thus

(3)

$$u_n(x,t)=X_n(x)T_n(t)=\sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)\cos\left(\left(\frac{2n+1}{2}\right)\pi t\right) \qquad (n=0,1,2,\ldots)$$

(21 pts. to here)

(21 pts. to here)

solves ①-②-③-④. The superposition principle then implies

(5 pts.) $$u(x,t) = \sum_{n=0}^{N} b_n \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right)\cos\left(\left(\frac{2n+1}{2}\right)\pi t\right)$$

solves ①-②-③-④ for any integer $N \geq 0$ and any constants $b_0, b_1, \ldots, b_N$.

Applying ⑤, we have

$$\frac{1}{2}\sin^3\left(\frac{\pi x}{2}\right) = u(x,0) = \sum_{n=0}^{N} b_n \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right) \quad \text{for } 0 \leq x \leq 1.$$

Using the identity $\sin^3(\theta) = \frac{3}{4}\sin(\theta) - \frac{1}{4}\sin(3\theta)$ with $\theta = \frac{\pi x}{2}$ gives

(1 pts.) $$\frac{3}{8}\sin\left(\frac{\pi x}{2}\right) - \frac{1}{8}\sin\left(\frac{3\pi x}{2}\right) = \sum_{n=0}^{N} b_n \sin\left(\left(\frac{2n+1}{2}\right)\pi x\right) \quad \text{for } 0 \leq x \leq 1.$$

(4 pts.) By inspection, we may satisfy this equation by taking $N=1$, $b_0 = \frac{3}{8}$, and $b_1 = -\frac{1}{8}$. Therefore

(4 pts.) $$\boxed{u(x,t) = \frac{3}{8}\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi t}{2}\right) - \frac{1}{8}\sin\left(\frac{3\pi x}{2}\right)\cos\left(\frac{3\pi t}{2}\right)}$$

solves ①-②-③-④-⑤.

(10 pts.) **Bonus**: Suppose $u = u_1(x,t)$ and $u = u_2(x,t)$ are solutions to ①-②-③-④-⑤.

(1 pt.) Then $w(x,t) = u_1(x,t) - u_2(x,t)$ solves

$$w_{tt} - w_{xx} \overset{⑪}{=} 0 \quad \text{if } 0 < x < 1,\ 0 < t < \infty,$$

$$w(0,t) \overset{⑫}{=} 0 \overset{⑬}{=} w_x(1,t) \quad \text{if } 0 \leq t < \infty,$$

$$w(x,0) \overset{⑭}{=} 0 \overset{⑮}{=} w_t(x,0) \quad \text{if } 0 \leq x \leq 1.$$

Let $E(t) = \int_0^1 \left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx \quad (0 \leq t < \infty)$ be the energy of the solution $w = w(x,t)$ to ⑪-⑫-⑬-⑭-⑮. Then

$$\frac{dE}{dt} = \int_0^1 \frac{\partial}{\partial t}\left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx$$

$$= \int_0^1 \left[w_t(x,t)w_{tt}(x,t) + w_x(x,t)w_{tx}(x,t)\right]dx.$$

Integrating by parts the second term in the integrand and applying ⑬ and the derivative of ⑫: $w_t(0,t) = 0$, gives

$$\frac{dE}{dt} = \int_0^1 w_t(x,t)w_{tt}(x,t)dx + \overbrace{w_x(1,t)}^{0}w_t(1,t) - w_x(0,t)\overbrace{w_t(0,t)}^{0} - \int_0^1 w_t(x,t)w_{xx}(x,t)dx$$

$$= \int_0^1 w_t(x,t)\left[w_{tt}(x,t) - w_{xx}(x,t)\right]dx .$$

Applying ⑪ then gives $\frac{dE}{dt} = 0$; i.e. the energy function is constant on the interval $0 \le t < \infty$. But then ⑮ and the derivative of ⑭: $w_x(x,0) = 0$, produce

$$0 \le E(t) = E(0) = \int_0^1 \left[\tfrac{1}{2}w_t^2(x,0) + \tfrac{1}{2}w_x^2(x,0)\right]dx = 0 .$$

The vanishing principle then gives $\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t) = 0$ for all $0 \le x \le 1$ and each $0 \le t < \infty$. But then $w_t(x,t) = 0 = w_x(x,t)$ for all $0 \le x \le 1$ and $0 \le t < \infty$ so $w(x,t)$ is constant on the strip $0 \le x \le 1$, $0 \le t < \infty$. Applying ⑭ yields $w(x,t) = 0$ on the strip $0 \le x \le 1$, $0 \le t < \infty$. That is, $u_1(x,t) = u_2(x,t)$ for all $0 \le x \le 1$ and $0 \le t < \infty$ so there is at most one solution to ①-②-③-④-⑤.

(By problem 2 of this exam, we know $u(x,t) = \frac{3}{8}\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi t}{2}\right) - \frac{1}{8}\sin\left(\frac{3\pi x}{2}\right)\cos\left(\frac{3\pi t}{2}\right)$ is a solution to ①-②-③-④-⑤, so this solution is unique.)

3.(40 pts.) (a) If f is an absolutely integrable function on $(-\infty,\infty)$ and b is a real number, show that the function $g(x)=f(x-b)$ has Fourier transform

$$\hat{g}(\xi)=e^{-ib\xi}\hat{f}(\xi).$$

(b) Use Fourier transform methods to find a formula for the solution to

$$u_t-u_{xx}+u_x \overset{①}{=} 0 \quad \text{if} \quad -\infty<x<\infty,\ 0<t<\infty,$$

subject to the initial condition $u(x,0)\overset{②}{=}\varphi(x)$ if $-\infty<x<\infty$. Let $y=x-b$.

5 (a) $$\hat{g}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-i\xi x}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-b)e^{-i\xi x}\,dx \overset{\downarrow}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i\xi(y+b)}\,dy$$

$$=\frac{e^{-ib\xi}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i\xi y}\,dy = e^{-ib\xi}\hat{f}(\xi).$$

35 (b) *Assume $u=u(x,t)$ is a solution to ① and take the Fourier transform with respect to x of both sides of ①:*

(2) $$\mathcal{F}(u_t-u_{xx}+u_x)(\xi)=\mathcal{F}(0)(\xi)$$

(7) $$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi)-(i\xi)^2\mathcal{F}(u)(\xi)+i\xi\mathcal{F}(u)(\xi)=0$$

(2) $$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi)+(\xi^2+i\xi)\mathcal{F}(u)(\xi)=0.$$

This is a linear, first-order, ordinary differential equation in the independent variable t; here ξ is a parameter. An integrating factor is

(3) $$\mu(t)=e^{\int(\xi^2+i\xi)dt}=e^{(\xi^2+i\xi)t+\cancel{c}^{\,0}}.$$

Multiplying both sides of the ODE by the integrating factor gives

(2) $$e^{(\xi^2+i\xi)t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi)+(\xi^2+i\xi)e^{(\xi^2+i\xi)t}\mathcal{F}(u)(\xi)=0.$$

But the left member of the equation is exact; i.e. it's the derivative of a single expression, namely

(2) $$\frac{\partial}{\partial t}\left[e^{(\xi^2+i\xi)t}\mathcal{F}(u)(\xi)\right]=0.$$

(18 pts. to here.)

(18 pts. to here)

Integrating both sides gives

(2) $$e^{(\xi^2+i\xi)t}\mathcal{F}(u)(\xi) = c_1(\xi).$$

Rearranging,

(2) (*) $$\mathcal{F}(u)(\xi) = e^{-i\xi t}c_1(\xi)e^{-\xi^2 t}.$$

(2) Set $t=0$ in the above equation and apply the initial condition ② to obtain $\mathcal{F}(\varphi)(\xi) = c_1(\xi)$. Also formula I in the table of Fourier transforms can be written as

$$\mathcal{F}\left(\sqrt{2a}\, e^{-a(\cdot)^2}\right)(\xi) = e^{-\frac{\xi^2}{4a}}.$$

Set $\frac{1}{4a}=t$ (or equivalently $\frac{1}{4t}=a$) in this formula to obtain

(3) $$\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi) = e^{-\xi^2 t}.$$

Substituting in (*) gives

(2) $$\mathcal{F}(u)(\xi) = e^{-i\xi t}\mathcal{F}(\varphi)(\xi)\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi).$$ (29 pts. to here)

Applying the convolution property of Fourier transforms produces

(2) $$\mathcal{F}(u)(\xi) = e^{-i\xi t}\frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi * \frac{1}{\sqrt{2t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi)$$

$$= e^{-i\xi t}\mathcal{F}\left(\varphi * \frac{1}{\sqrt{4\pi t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi).$$

By the inversion theorem and part (a) of this problem with $b=t$ yields

(2) $$u(x,t) = \left(\varphi * \frac{1}{\sqrt{4\pi t}}e^{-\frac{(\cdot)^2}{4t}}\right)(x-t)$$

(2) $$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-t-y)^2}{4t}}\varphi(y)\,dy}.$$ (35 pts. to here)

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$
A.	$\begin{cases}1 & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases}1 & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2+a^2} \quad (a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases}x & \text{if } 0<x\le b,\\ 2b-x & \text{if } b<x<2b,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases}e^{-ax} & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases}e^{ax} & \text{if } b<x<c,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases}e^{iax} & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases}e^{iax} & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a>0)$	$\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a>0)$	$\begin{cases}0 & \text{if } \lvert\xi\rvert\ge a,\\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert<a.\end{cases}$

Math 325

Exam II

Summer 2013

number of exams: 8

mean: 76.5

median: 82.0

standard deviation: 22.0

Distribution of Scores

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87–110	A	A	3
73 – 86	B	B	2
60 – 72	C	B	0
50 – 59	C	C	2
0 – 49	F	D	1