Mathematics 325 Exam III Name: Dr. Grow

Summer 2014 (1 pt.)

1.(33 pts.) (a) Show that $\Phi=\{\varphi_n(x)\}_{n=1}^{\infty}=\left\{\sin\left(\frac{(2n-1)\pi x}{2}\right)\right\}_{n=1}^{\infty}$ is an orthogonal set of functions on the interval $[0,1]$. (You may find the identity $\sin(A)\sin(B)=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$ useful.)

(b) Compute the Fourier coefficients $B_n(f)=\frac{\langle f,\varphi_n\rangle}{\langle\varphi_n,\varphi_n\rangle}$ of $f(x)=x(2-x)$ with respect to the orthogonal set Φ on $[0,1]$.

(c) Show that the Fourier series of $f(x)=x(2-x)$ with respect to Φ on $[0,1]$ is given by

$$\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi x/2)}{(2n-1)^3}.$$

(d) Assuming that the Fourier series of f at x converges to $f(x)$ for all x in $[0,1]$, find the sum of

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}.$$

8 pts. (a) Let m and n be distinct positive integers. Then

$$\langle\varphi_n,\varphi_m\rangle=\int_0^1\varphi_n(x)\overline{\varphi_m(x)}\,dx=\int_0^1\sin\left(\frac{(2n-1)\pi x}{2}\right)\sin\left(\frac{(2m-1)\pi x}{2}\right)dx=\frac{1}{2}\int_0^1\left[\cos\left(\frac{2(n-m)\pi x}{2}\right)-\cos\left(\frac{2(m+n-1)\pi x}{2}\right)\right]dx$$

$$=\frac{1}{2}\left[\frac{\sin((n-m)\pi x)}{(n-m)\pi}-\frac{\sin((m+n-1)\pi x)}{(m+n-1)\pi}\right]\Bigg|_{x=0}^{1}=\boxed{0.}$$

Also, note that when m=n, the above calculation gives $\langle\varphi_n,\varphi_n\rangle=\frac{1}{2}\int_0^1[1-\cos((2n-1)\pi x)]dx$

$$=\frac{1}{2}\left[x-\frac{\sin((2n-1)\pi x)}{(2n-1)\pi}\right]\Bigg|_{x=0}^{1}=\boxed{\frac{1}{2}.}$$

8 pts. (b) $B_n(f)=\frac{\langle f,\varphi_n\rangle}{\langle\varphi_n,\varphi_n\rangle}=\frac{\int_0^1 f(x)\overline{\varphi_n(x)}\,dx}{1/2}=2\int_0^1 \overbrace{x(2-x)}^{u}\overbrace{\sin\left(\frac{(2n-1)\pi x}{2}\right)dx}^{dv}\overset{\text{Parts}}{=}$

$$-2x(2-x)\frac{2}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{2}\right)\Bigg|_{x=0}^{1}\to 0-2\int_0^1\frac{-2}{(2n-1)\pi}\overbrace{(2-2x)}^{u}\overbrace{\cos\left(\frac{(2n-1)\pi x}{2}\right)dx}^{dv}\overset{\text{Parts}}{=}$$

$$\frac{4}{(2n-1)\pi}(2-2x)\frac{2}{(2n-1)\pi}\sin\left(\frac{(2n-1)\pi x}{2}\right)\Bigg|_0^1\to 0-\frac{4}{(2n-1)\pi}\int_0^1\frac{2}{(2n-1)\pi}\sin\left(\frac{(2n-1)\pi x}{2}\right)(-2dx)=\frac{16}{(2n-1)^2\pi^2}\frac{-2}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{2}\right)\Bigg|_0^1$$

$$=\boxed{\frac{32}{(2n-1)^3\pi^3}}\quad(n=1,2,3,\ldots)$$

(c) The Fourier series of $f(x) = x(2-x)$ with respect to Φ on $[0,1]$ is

8 pts.

$$\sum_{n=1}^{\infty} B_n(f)\varphi_n(x) = \sum_{n=1}^{\infty} \frac{32}{(2n-1)^3\pi^3} \sin\left(\frac{(2n-1)\pi x}{2}\right)$$

$$= \boxed{\frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)}{(2n-1)^3}}$$

(d) Assume

9 pts.

$$f(x) = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)}{(2n-1)^3}$$

for all x in $[0,1]$. Taking $x=1$, we have

$$1 = 1(2-1) = f(1) = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi/2)}{(2n-1)^3},$$

or equivalently,

$$\boxed{\frac{\pi^3}{32}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}.$$

n	$\sin\left(\frac{(2n-1)\pi}{2}\right)$
1	$\sin\frac{\pi}{2} = 1$
2	$\sin\frac{3\pi}{2} = -1$
3	$\sin\left(\frac{5\pi}{2}\right) = 1$
⋮	⋮
n	$(-1)^{n-1}$

2.(33 pts.) Solve $u_{tt} - u_{xx} \overset{①}{=} 0$ in the strip $0 < x < 1,\ 0 < t < \infty$, subject to the boundary conditions $u(0,t) \overset{②}{=} 0 \overset{③}{=} u_x(1,t)$ if $t \geq 0$ and the initial conditions $u(x,0) \overset{⑤}{=} x(2-x)$ and $u_t(x,0) \overset{④}{=} 0$ if $0 \leq x \leq 1$. (You may find the results of problem 1 useful.)

1 pt. We use separation of variables. We seek nontrivial solutions of the form $u(x,t) = X(x)T(t)$ to the homogeneous portion of the problem: ①-②-③-④. Applying ① yields

3 pts. to here $X(x)T''(t) - X''(x)T(t) = 0$ for all $0<x<1,\ 0<t<\infty$. Rearranging this leads to

4 $$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} = \text{constant} = \lambda.$$ Applying ②,③, and ④ gives

$X(0)T(t) = 0 = X'(1)T(t)$ for all $t \geq 0$ and $X(x)T'(0) = 0$ for all $0 \leq x \leq 1$.

7 pts. to here Since $u(x,t) = X(x)T(t)$ is not identically zero on $0 \leq x \leq 1,\ 0 \leq t < \infty$, we must have $X(0) = 0 = X'(1)$ and $T'(0) = 0$. Therefore we are led to solve the coupled system

11 pts. to here
13 pts. to here
$$\begin{cases} X''(x) + \lambda X(x) \overset{⑥}{=} 0, & X(0) \overset{⑦}{=} 0 \overset{⑧}{=} X'(1) \\ T''(t) + \lambda T(t) \overset{⑨}{=} 0, & T'(0) \overset{⑩}{=} 0 \end{cases}$$

Note that ⑥-⑦-⑧ is an eigenvalue problem. We assume that the eigenvalues are real.

<u>Case $\lambda > 0$</u> (say $\lambda = \alpha^2$ where $\alpha > 0$): Then ⑥ becomes $X''(x) + \alpha^2 X(x) = 0$ with general solution $X(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x)$ and derivative $X'(x) = -c_1\alpha\sin(\alpha x) + c_2\alpha\cos(\alpha x)$. Applying ⑦ yields $0 = X(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1$. Applying ⑧ gives $0 = X'(1) = -\cancelto{0}{c_1}\alpha\sin(\alpha) + c_2\alpha\cos(\alpha)$ so $\cos(\alpha) = 0$ is necessary to obtain a nontrivial solution. Then

$\alpha = \alpha_n = \frac{(2n-1)\pi}{2}$ where $n = 1,2,3,\ldots$ Hence we have:

16 pts. to here eigenvalues: $\lambda_n = \alpha_n^2 = \left(\frac{(2n-1)\pi}{2}\right)^2$,

19 pts. to here eigenfunctions: $X_n(x) = \sin(\alpha_n x) = \sin\left(\frac{(2n-1)\pi x}{2}\right)$. $\Big\}\ n = 1,2,3,\ldots$

<u>Case $\lambda = 0$</u>: Then ⑥ becomes $X''(x) = 0$ with general solution $X(x) = c_1 x + c_2$ and derivative $X'(x) = c_1$. Applying ⑦ and ⑧ yields

20 $0 = X(0) = c_2$ and $0 = X'(1) = c_1$ so all solutions are trivial in this case.

Case $\lambda<0$ (say $\lambda=-\alpha^2$ where $\alpha>0$): Then ⑥ becomes $X''(x)-\alpha^2X(x)=0$ with general solution $X(x)=c_1\cosh(\alpha x)+c_2\sinh(\alpha x)$ and derivative $X'(x)=\alpha c_1\sinh(\alpha x)+\alpha c_2\cosh(\alpha x)$. Applying ⑦ gives $0=X(0)=c_1\cdot 1+c_2\cdot 0=c_1$ and ⑧ gives $0=X'(1)=\alpha \overset{0}{\cancel{c_1}}\sinh(\alpha)+\alpha c_2\cosh(\alpha)$
21 pts. to here so $c_2=0$ (since $\alpha\cosh\alpha>0$ when $\alpha>0$). Therefore all solutions are trivial in this case.

When $\lambda=\lambda_n=\alpha_n^2=\left(\frac{(2n-1)\pi}{2}\right)^2$, ⑨-⑩ becomes $T_n''(t)+\alpha_n^2T_n(t)\overset{(9')}{=}0$, $T_n'(0)\overset{(10')}{=}0$. The general solution of (9') is $T_n(t)=c_1\cos(\alpha_n t)+c_2\sin(\alpha_n t)$ with derivative $T_n'(t)=-\alpha_n c_1\sin(\alpha_n t)+\alpha_n c_2\cos(\alpha_n t)$.
24 Applying (10') gives $0=T_n'(0)=-\alpha_n c_1\cdot 0+\alpha_n c_2\cdot 1=c_2\alpha_n$ so $c_2=0$ and $T_n(t)=\cos(\alpha_n t)$, up to a constant factor. Therefore

$$u_n(x,t)=X_n(x)T_n(t)=\sin(\alpha_n x)\cos(\alpha_n t)=\sin\left(\frac{(2n-1)\pi x}{2}\right)\cos\left(\frac{(2n-1)\pi t}{2}\right)$$

25 pts. to here

solves ①-②-③-④ for each $n=1,2,3,\ldots$ By the superposition principle

28
$$u(x,t)=\sum_{n=1}^{\infty}c_n\sin\left(\frac{(2n-1)\pi x}{2}\right)\cos\left(\frac{(2n-1)\pi t}{2}\right)$$

is a "formal" solution of ①-②-③-④ for "any" choice of constants $c_1,c_2,c_3,\ldots$ In order to satisfy ⑤ we must have

$$f(x)=x(2-x)=u(x,0)=\sum_{n=1}^{\infty}c_n\sin\left(\frac{(2n-1)\pi x}{2}\right)$$

for all $0\le x\le 1$. Therefore, by problem 1, we must choose $c_n=B_n(f)=\frac{32}{(2n-1)^3\pi^3}$ for $n=1,2,3,\ldots$ Consequently,

$$\boxed{u(x,t)=\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi x/2)\cos((2n-1)\pi t/2)}{(2n-1)^3}}$$

33 pts. to here.

solves ①-②-③-④-⑤.

3.(33 pts.) (a) Show that the solution to problem 2 on this exam is unique.
(b) Is the solution to problem 2 on this exam periodic in time? That is, does there exist a number $T > 0$ such that $u(x,t+T) = u(x,t)$ for all $0 \le x \le 1$ and all $t \ge 0$? If not, explain why not. If so, calculate the smallest positive period of the solution.
(c) Is the solution to problem 2 on this exam bounded? If not, explain why not. If so, find the maximum and minimum values of the solution.

(a) Yes, the solution to problem 2 on this exam is unique. To see why this is true, we use the energy method. Suppose $u = v(x,t)$ is another solution to problem 2 and define

1 pt. $w(x,t) = u(x,t) - v(x,t)$ for $0 \le x \le 1$, $0 \le t < \infty$, where $u = u(x,t)$ is the solution we obtained for problem 2. Then w satisfies

$$w_{tt} - w_{xx} \overset{①}{=} 0 \quad \text{if } 0 < x < 1,\ 0 < t < \infty,$$

3 pts. to here.

$$w(0,t) \overset{②}{=} 0 \overset{③}{=} w_x(1,t) \quad \text{if } t \ge 0,$$

$$w(x,0) \overset{⑤}{=} 0 \overset{④}{=} w_t(x,0) \quad \text{if } 0 \le x \le 1.$$

5 pts. to here. Let $E(t) = \int_0^1 \left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx$ denote the energy function for w on $0 \le t < \infty$.

7 Then

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^1 \left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx = \int_0^1 \frac{\partial}{\partial t}\left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx = \int_0^1 \Big[w_t(x,t)w_{tt}(x,t) +$$

8

$$\overbrace{w_x(x,t)}^{v}\overbrace{w_{xt}(x,t)}^{dv}\Big]dx \overset{\text{Parts}}{=} \int_0^1 w_t(x,t)w_{tt}(x,t)dx + w_x(x,t)w_t(x,t)\Big|_{x=0}^{1} - \int_0^1 w_t(x,t)w_{xx}(x,t)dx.$$

But ② implies $w_x(0,t) = \lim_{h\to 0} \frac{w(h,t) - w(0,t)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$ so

9 pts. to here.

$$w_x(x,t)w_t(x,t)\Big|_{x=0}^{1} = w_x(1,t)w_t(1,t) - w_x(0,t)w_t(0,t) = 0\cdot w_t(1,t) - 0\cdot w_t(0,t) = 0 \text{ by ③}.$$

11 Therefore $\frac{dE}{dt} = \int_0^1 w_t(x,t)\big[\overbrace{w_{tt}(x,t) - w_{xx}(x,t)}^{0 \text{ by ①}}\big]dx = 0$ and thus, for $t \ge 0$,

13

$$E(t) = E(0) = \int_0^1 \left[\frac{1}{2}w_t^2(x,0) + \frac{1}{2}w_x^2(x,0)\right]dx = \int_0^1 \left[\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0\right]dx = 0$$

by ④ and ⑤. $\left(w_x(x,0) = \lim_{h\to 0} \frac{w(x+h,0) - w(x,0)}{h} \overset{⑤}{=} \lim_{h \to 0} \frac{0-0}{h} = 0. \right)$

15 pts. to here. Therefore, the vanishing theorem implies $\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t) = 0$ for $0 \le x \le 1$ and each fixed $t \ge 0$. Thus $w_t(x,t) = 0 = w_x(x,t)$ for all $0 \le x \le 1$ and all $0 \le t < \infty$, and

16 it follows that $w(x,t) = $ constant on $0 \le x \le 1$, $0 \le t < \infty$. But ⑤ implies this

constant is equal to zero. That is, $0 = w(x,t) = u(x,t) - v(x,t)$ if $0 \le x \le 1,\ 0 \le t < \infty$.
This shows that $v(x,t) = u(x,t)$ on $0 \le x \le 1,\ 0 \le t < \infty$, proving uniqueness of the solution $u = u(x,t)$ to problem 2.

(b) Yes, the solution to problem 2 is periodic in the variable t with smallest positive period $T = 4$. To see this, note that

$$u(x,t+4) = \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)\cos((2n-1)\pi(t+4)/2)}{(2n-1)^3} = \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)\cos((2n-1)\pi t/2 + 2(2n-1)\pi)}{(2n-1)^3}$$

$$= \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)\cos((2n-1)\pi t/2)}{(2n-1)^3} = u(x,t)$$

for all $0 \le x \le 1,\ 0 \le t < \infty$; i.e. $T = 4$ is a period of the solution to problem 2 in time.

5 pts. to here.

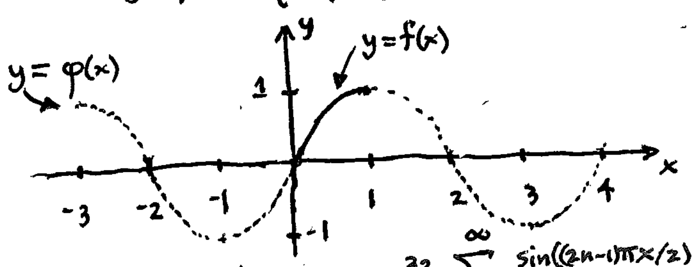
Let T_1 be the smallest positive period in time for the solution $u = u(x,t)$ to problem 2. Then all positive periods in time for u are integer multiples of T_1 so $mT_1 = 4$ for some integer $m \ge 1$. Then for every integer $n \ge 1$ and $t \ge 0$,

$$\cos\left(\frac{(2n-1)\pi t}{2}\right) = \cos\left((2n-1)\frac{\pi}{2}\left(t + \frac{4}{m}\right)\right) = \cos\left(\frac{(2n-1)\pi t}{2}\right)\cos\left(\frac{(2n-1)2\pi}{m}\right) - \sin\left(\frac{(2n-1)\pi t}{2}\right)\sin\left(\frac{(2n-1)2\pi}{m}\right)$$

so $\frac{(2n-1)2\pi}{m} = 2k_n\pi$ for some positive integer k_n. In particular, when $n=1$, we have $1 \ge \frac{1}{m} = k_1 \in \mathbb{N}$ and $m = 1$ follows; i.e. $T_1 = 4$.

8 pts. to here.

(c) Yes, the solution $u = u(x,t)$ to problem 2 is bounded on $0 \le x \le 1,\ 0 \le t < \infty$, with $u_{max} = 1$ and $u_{min} = -1$. To see this, note that the function f in problems 1 and 2 can be extended to the entire real line by $\varphi(x) = \begin{cases} x(2-x) & \text{if } 0 \le x < 2, \\ -x(2+x) & \text{if } -2 \le x < 0, \end{cases}$ and $\varphi(x+4) = \varphi(x)$ for all real x.



Furthermore $-1 \le \varphi(x) \le 1$ and $\varphi(x) = \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)}{(2n-1)^3}$ for all $-\infty < x < \infty$.

Applying the identity $\sin(A)\cos(B) = \frac{1}{2}(\sin(A-B) + \sin(A+B))$, we have

$$u(x,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)\cos((2n-1)\pi t/2)}{(2n-1)^3} = \frac{16}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\frac{\pi}{2}(x-t))}{(2n-1)^3} + \frac{16}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\frac{\pi}{2}(x+t))}{(2n-1)^3}$$

so $u(x,t) = \frac{1}{2}\varphi(x-t) + \frac{1}{2}\varphi(x+t)$ for all $0 \le x \le 1$ and $0 \le t < \infty$. Consequently,

6 pts. to here.

$$-1 = \frac{1}{2}\varphi_{min} + \frac{1}{2}\varphi_{min} \le \overbrace{\frac{1}{2}\varphi(x-t) + \frac{1}{2}\varphi(x+t)}^{u(x,t)} \le \frac{1}{2}\varphi_{max} + \frac{1}{2}\varphi_{max} = 1$$

for all $0 \le x \le 1$ and $0 \le t < \infty$. But $u(1,0) = f(1) = \varphi(1) = 1$ and

$$u(1,2) = \frac{1}{2}\varphi(1-2) + \frac{1}{2}\varphi(1+2) = \frac{1}{2}\varphi(-1) + \frac{1}{2}\varphi(3-4) = \varphi(-1) = -\varphi(1) = -1.$$

8 pts. to here.

Consequently, $u_{max} = 1$ and $u_{min} = -1$.

Math 325

Exam III

Summer 2014

$\mu = 73.0$

$\sigma = 14.3$

$n = 14$

Range	Grad. Letter Grade	Undergrad. Letter Grade	Frequency
87–100	A	A	4
73–86	B	B	2
60–72	C	B	5
50–59	C	C	3
0–49	F	D	0