Mathematics 325 | Final Exam Summer 2014 | Name: Dr. Grow

1.(25 pts.) (a) Show that $u(x,y)=(x+y)^3$ is a solution of $yu_x - xu_y = 3(y-x)(x+y)^2$ in the $xy-$plane.

(b) Find the general solution of $yu_x - xu_y = 3(y-x)(x+y)^2$ in the $xy-$plane.

2.(25 pts.) Classify the partial differential equation $u_{xx}+4u_{yy}-4u_{xy}+25u=0$ as elliptic, parabolic, or hyperbolic and find the general solution in the $xy-$plane, if possible.

3.(25 pts.) A homogeneous solid material occupies the region in a unit cube. In the interior of the cube, heat exchange takes place according to Fourier's law - the flow is from hot to cold regions and the velocity of the flow is proportional to the temperature gradient. On the faces of the cube, heat exchange takes place according to Newton's law of cooling - the normal component of the heat flux is proportional to the temperature difference. Assume that the cube is initially uniformly at temperature 100 degrees Celsius and is surrounded by a medium of constant temperature 20 degrees Celsius. Write - no proof or derivation is necessary - the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) and time t in the cube.

4.(25 pts.) Use Fourier transform methods to find a formula for the solution to $u_t - ku_{xx} = f(x,t)$ in the upper half-plane $-\infty < x < \infty,\ 0 < t < \infty$, subject to the initial condition $u(x,0)=0$ if $-\infty < x < \infty$.

5.(25 pts.) (a) Show that the Fourier series of the function $f(x)=1-x^2$ with respect to the orthogonal set of functions $\Phi=\{\varphi_n(x)=\cos((2n+1)\pi x/2):\ n=0,1,2,\dots\}$ on the interval $[0,1]$ is given by

$$\sum_{n=0}^{\infty}\frac{32(-1)^n\cos((2n+1)\pi x/2)}{(2n+1)^3\pi^3}.$$

(b) Discuss the convergence, or lack thereof, for the Fourier series of f with respect to Φ on $[0,1]$. Justify your answers for each of the three types of convergence - uniform, L^2, and pointwise - using the theorems at the end of this exam.

6.(25 pts.) Solve $u_{tt}-u_{xx}=0$ on $0<x<1,\ 0<t<\infty$, subject to $u_x(0,t)=0=u(1,t)$ if $t\ge 0$ and $u(x,0)=1-x^2,\ u_t(x,0)=0$ if $0\le x\le 1$. You may find the results of problem 5 useful.

7.(25 pts.) Solve the Poisson equation $\nabla^2 u = 1$ in the spherical shell $1<r<3$, given that $u=0$ on $r=1$ and $\dfrac{\partial u}{\partial r}=0$ on $r=3$. You may find useful the fact that the three-dimensional Laplacian in spherical polar coordinates is $\nabla^2 u = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)+\dfrac{1}{r^2\sin(\phi)}\dfrac{\partial}{\partial\phi}\left(\sin(\phi)\dfrac{\partial u}{\partial\phi}\right)+\dfrac{1}{r^2\sin^2(\phi)}\dfrac{\partial^2 u}{\partial\theta^2}$.

8.(25 pts.) Solve $u_t-u_{xx}-u_{yy}=0$ for $0<x<\pi,\ 0<y<\pi,\ 0<t<\infty$, given that for all times $t\ge 0$ the solution satisfies homogeneous Neumann boundary conditions: $\dfrac{\partial u}{\partial n}=0$ for $x=0,\ x=\pi,\ y=0,$ and $y=\pi$, and the initial condition $u(x,y,0)=\cos^2(x)\cos^2(y)$ for $0\le x\le\pi,\ 0\le y\le\pi$.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$
A.	$\begin{cases}1 & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases}1 & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2+a^2}\quad (a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases}x & \text{if } 0<x\le b,\\ 2b-x & \text{if } b<x<2b,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases}e^{-ax} & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases}e^{ax} & \text{if } b<x<c,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases}e^{iax} & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases}e^{iax} & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2}\quad (a>0)$	$\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x}\quad (a>0)$	$\begin{cases}0 & \text{if } \lvert\xi\rvert\ge a,\\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert<a.\end{cases}$

Fourier Series Convergence Theorems

Consider the eigenvalue problem

(1) $$X''(x)+\lambda X(x)=0 \text{ in } a<x<b$$

with any symmetric boundary conditions of the form

(2) $$\begin{cases} \alpha_1 f(a)+\beta_1 f(b)+\gamma_1 f'(a)+\delta_1 f'(b)=0 \\ \alpha_2 f(a)+\beta_2 f(b)+\gamma_2 f'(a)+\delta_2 f'(b)=0 \end{cases}$$

and let $\Phi=\{X_1,X_2,X_3,\ldots\}$ be the complete orthogonal set of eigenfunctions for (1)-(2). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$\sum_{n=1}^{\infty} A_n X_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle} \quad (n=1,2,3,\ldots).$$

Theorem 2. (Uniform Convergence) If

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and

(ii) f satisfies the given symmetric boundary conditions,

then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 – Convergence) If

$$\int_a^b |f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)

(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.

(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty} A_n X_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x.

#1.

(a) If $u(x,y)=(x+y)^3$ then

$$yu_x - xu_y = y\left(3(x+y)^2\right) - x\left(3(x+y)^2\right) \overset{\checkmark}{=} 3(y-x)(x+y)^2.$$

(b) The general solution to the nonhomogeneous PDE is of the form $u = u_0 + u_p$ where $u_0 = u_0(x,y)$ is the general solution of the associated homogeneous equation $yu_x - xu_y = 0$ and $u_p = u_p(x,y) = (x+y)^3$ is a particular solution to the nonhomogeneous PDE. The characteristic curves for $a(x,y)u_x + b(x,y)u_y = 0$ are given by $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$. Therefore $\frac{dy}{dx} = \frac{-x}{y}$ describes the characteristic curves of $yu_x - xu_y = 0$. Separating variables and integrating gives

$$\frac{y^2}{2} = \int y\,dy = \int -x\,dx = -\frac{x^2}{2} + c_1 \quad \Rightarrow \quad x^2 + y^2 = c \qquad (c = 2c_1).$$

Along such a characteristic curve, $u_0(x, y(x)) = u(0, y(0)) = u(0, \pm\sqrt{c}) = f(c)$.
Therefore $u_0(x,y) = f(x^2+y^2)$ where $f \in C^1(\mathbb{R})$. Thus

$$\boxed{u(x,y) = f(x^2+y^2) + (x+y)^3}$$

is the general solution of $yu_x - xu_y = 3(y-x)(x+y)^2$ where f is an arbitrary continuously differentiable function of a single real variable.

2.

3pts. to here. Since $B^2-4AC=(-4)^2-4(1)(4)=0$, the PDE is parabolic.

To solve it, we rewrite it as $\left(\frac{\partial^2}{\partial x^2}-4\frac{\partial^2}{\partial x\partial y}+4\frac{\partial^2}{\partial y^2}\right)u+25u=0$, or equivalently,

5 $\left(\frac{\partial}{\partial x}-2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-2\frac{\partial}{\partial y}\right)u+25u=0$. This suggests the change-of-coordinates

9 $\begin{cases}\xi = x-2y,\\ \eta = 2x+y.\end{cases}$ Then the chain rule implies $\frac{\partial v}{\partial x}=\frac{\partial v}{\partial \xi}\overset{1}{\cancel{\frac{\partial \xi}{\partial x}}}+\frac{\partial v}{\partial \eta}\overset{2}{\cancel{\frac{\partial \eta}{\partial x}}}=\frac{\partial v}{\partial \xi}+2\frac{\partial v}{\partial \eta}$

and $\frac{\partial v}{\partial y}=\frac{\partial v}{\partial \xi}\overset{-2}{\cancel{\frac{\partial \xi}{\partial y}}}+\frac{\partial v}{\partial \eta}\overset{1}{\cancel{\frac{\partial \eta}{\partial y}}}=-2\frac{\partial v}{\partial \xi}+\frac{\partial v}{\partial \eta}$. That is, as operators we have

13 $\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}+2\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y}=\frac{\partial}{\partial \eta}-2\frac{\partial}{\partial \xi}$. Substituting in (b) gives

15 $$\left[\left(\frac{\partial}{\partial \xi}+\cancel{2\frac{\partial}{\partial \eta}}\right)-2\left(\cancel{\frac{\partial}{\partial \eta}}-2\frac{\partial}{\partial \xi}\right)\right]\left[\left(\frac{\partial}{\partial \xi}+\cancel{2\frac{\partial}{\partial \eta}}\right)-2\left(\cancel{\frac{\partial}{\partial \eta}}-2\frac{\partial}{\partial \xi}\right)\right]u+25u=0$$

$$\left(5\frac{\partial}{\partial \xi}\right)\left(5\frac{\partial}{\partial \xi}\right)u+25u=0$$

17 $$\frac{\partial^2 u}{\partial \xi^2}+u=0.$$

20 The general solution of this PDE is $u=c_1(\eta)\cos(\xi)+c_2(\eta)\sin(\xi)$. Therefore

25 pts. to here $$\boxed{u(x,y)=f(2x+y)\cos(x-2y)+g(2x+y)\sin(x-2y)}$$

is the general solution of (b) where f and g are arbitrary, twice-continuously differentiable functions of a single real variable.

#3. In the interior of the cube only conductive heat transfer occurs, so the heat equation, $u_t - k\nabla^2 u = 0$, governs the changes in temperature at interior points of the cube. On the faces of the cube, the normal component of the heat flux $\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n}$ is a constant multiple of $u - T_0 = u - 20$. Initially $u(x,y,z,0) = 100$ at all points (x,y,z) of the cube. Therefore

10 pts.

10 pts.

5 pts.

$$
\begin{cases}
u_t(x,y,z,t) - k\left(\frac{\partial^2 u}{\partial x^2}(x,y,z,t) + \frac{\partial^2 u}{\partial y^2}(x,y,z,t) + \frac{\partial^2 u}{\partial z^2}(x,y,z,t)\right) = 0 \\
\qquad \text{if } 0<x<1, 0<y<1, 0<z<1, \text{ and } t>0; \\
u_x(0,y,z,t) = -c\big(u(0,y,z,t) - 20\big) \text{ and } u_x(1,y,z,t) = -c\big(u(1,y,z,t) - 20\big) \\
\qquad \text{if } 0\le y\le 1, 0\le z\le 1, \text{ and } t\ge 0; \\
u_y(x,0,z,t) = -c\big(u(x,0,z,t) - 20\big) \text{ and } u_y(x,1,z,t) = -c\big(u(x,1,z,t) - 20\big) \\
\qquad \text{if } 0\le x\le 1, 0\le z\le t, \text{ and } t\ge 0; \\
u_z(x,y,0,t) = -c\big(u(x,y,0,t) - 20\big) \text{ and } u_z(x,y,1,t) = -c\big(u(x,y,1,t) - 20\big) \\
\qquad \text{if } 0\le x\le 1, 0\le y\le 1, \text{ and } t\ge 0; \\
u(x,y,z,0) = 100 \quad \text{if } 0\le x\le 1, 0\le y\le 1, 0\le z\le 1.
\end{cases}
$$

models the temperature u at all points in the unit cube: $0\le x\le 1, 0\le y\le 1, 0\le z\le 1$ and at times $t \ge 0$. Here c and k are positive constants.

#4.

Suppose $u = u(x,t)$ is a solution to the problem. Then

(*) $\quad u_t(x,t) - k u_{xx}(x,t) = f(x,t)$

for all (x,t) in the upper half-plane. Taking the Fourier transform of (*) with respect to x holding t fixed gives

1 $$\mathcal{F}\big(u_t(\cdot,t) - k u_{xx}(\cdot,t)\big)(\xi) = \mathcal{F}\big(f(\cdot,t)\big)(\xi)$$

for all $-\infty < \xi < \infty$ so

$$\frac{\partial}{\partial t}\mathcal{F}\big(u(\cdot,t)\big)(\xi) - k(i\xi)^2\mathcal{F}\big(u(\cdot,t)\big)(\xi) = \mathcal{F}\big(f(\cdot,t)\big)(\xi)$$

3 (**) $$\frac{\partial \mathcal{F}}{\partial t}\big(u(\cdot,t)\big)(\xi) + k\xi^2\mathcal{F}\big(u(\cdot,t)\big)(\xi) = \mathcal{F}\big(f(\cdot,t)\big)(\xi)$$

This is a first order linear ODE in the independent variable t

4 (with parameter ξ) so an integrating factor is $\mu(t) = e^{\int k\xi^2 dt} = e^{k\xi^2 t}$.

Multiplying through (**) by the integrating factor and simplifying yields

6 $$\frac{\partial}{\partial t}\left[e^{k\xi^2 t}\mathcal{F}\big(u(\cdot,t)\big)(\xi)\right] = e^{k\xi^2 t}\frac{\partial \mathcal{F}}{\partial t}\big(u(\cdot,t)\big)(\xi) + k\xi^2 e^{k\xi^2 t}\mathcal{F}\big(u(\cdot,t)\big)(\xi) = e^{k\xi^2 t}\mathcal{F}\big(f(\cdot,$$

and integrating with respect to t holding ξ fixed,

8 $$e^{k\xi^2 t}\mathcal{F}\big(u(\cdot,t)\big)(\xi) = \int_0^t e^{k\xi^2\tau}\mathcal{F}\big(f(\cdot,\tau)\big)(\xi)\,d\tau + c(\xi).$$

Evaluating at $t=0$ and using the initial condition gives

9 $$0 = \mathcal{F}(0)(\xi) = \mathcal{F}\big(u(\cdot,0)\big)(\xi) = c(\xi) \qquad (-\infty<\xi<\infty).$$

Therefore $$\mathcal{F}\big(u(\cdot,t)\big)(\xi) = e^{-k\xi^2 t}\int_0^t e^{k\xi^2\tau}\mathcal{F}\big(f(\cdot,\tau)\big)(\xi)\,d\tau$$

11 pts to here. (***) $$= \int_0^t e^{-k\xi^2(t-\tau)}\mathcal{F}\big(f(\cdot,\tau)\big)(\xi)\,d\tau.$$

Applying formula I in the table of Fourier transforms with $a = \frac{1}{4k(t-\tau)}$

gives $\mathcal{F}\left(\sqrt{2a}\, e^{-a(\cdot)^2}\right)(\xi) = e^{-\frac{\xi^2}{4a}}$ so

13 pts. to here
$$\mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}}\, e^{-\frac{1}{4k(t-\tau)}(\cdot)^2}\right)(\xi) = e^{-k\xi^2(t-\tau)}.$$

Substituting in (***) and using the convolution property $\mathcal{F}(f*g)(\xi) = \sqrt{2\pi}\,\mathcal{F}(f)(\xi)\,\mathcal{F}(g)$

leads to

15
$$\mathcal{F}(u(\cdot,t))(\xi) = \int_0^t \mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}}\, e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\right)(\xi)\,\mathcal{F}(f(\cdot,\tau))(\xi)\,d\tau$$

17
$$= \int_0^t \frac{1}{\sqrt{2\pi}}\,\mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}}\, e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\right)(\xi)\,d\tau.$$

Interchanging the order of integration with respect to τ and x (in the Fourier transform) produces

19
$$\mathcal{F}(u(\cdot,t))(\xi) = \mathcal{F}\left(\int_0^t \frac{1}{\sqrt{4k\pi(t-\tau)}}\, e^{\frac{-(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\,d\tau\right)(\xi).$$

By the Fourier inversion formula and smoothness of the functions involved, we have

22
$$u(x,t) = \int_0^t \left(\frac{1}{\sqrt{4k\pi(t-\tau)}}\, e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\right)(x)\,d\tau$$

25 pts. to here
$$= \boxed{\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(t-\tau)}}\, e^{-\frac{(x-y)^2}{4k(t-\tau)}} f(y,\tau)\,dy\,d\tau}.$$

#5. (a) $$A_n(f) = \frac{\langle f, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle} = \frac{\int_0^1 \overbrace{(1-x^2)}^{u}\overbrace{\cos((2n+1)\pi x/2)\,dx}^{dv}}{\int_0^1 \cos^2((2n+1)\pi x/2)\,dx}.$$

$$\langle \varphi_n, \varphi_n\rangle = \int_0^1 \tfrac{1}{2}\left[1+\cos((2n+1)\pi x)\right]dx = \frac{x}{2} + \frac{\sin((2n+1)\pi x)}{(2n+1)\pi}\Bigg|_{x=0}^{1} = \frac{1}{2}.$$

$$\langle f, \varphi_n\rangle = \cancel{(1-x^2)\frac{\sin((2n+1)\pi x/2)}{(2n+1)\frac{\pi}{2}}\Bigg|_{x=0}^{1}}^{\,0} - \int_0^1 \frac{\sin((2n+1)\pi x/2)}{(2n+1)\frac{\pi}{2}}(-2x)\,dx$$

$$= \frac{4}{(2n+1)\pi}\int_0^1 \overbrace{x}^{u}\,\overbrace{\sin\left(\frac{(2n+1)\pi x}{2}\right)dx}^{dv} = \frac{4}{(2n+1)\pi}\left[\cancel{\frac{-2x}{(2n+1)\pi}\cos\left(\frac{(2n+1)\pi x}{2}\right)\Bigg|_0^1}^{\,0} - \int_0^1 \frac{-2}{(2n+1)\pi}\cos\left(\frac{(2n+1)\pi x}{2}\right)dx\right]$$

$$= \frac{8}{(2n+1)^2\pi^2}\cdot\frac{2}{(2n+1)\pi}\sin\left(\frac{(2n+1)\pi x}{2}\right)\Bigg|_0^1 = \frac{16\sin\left((2n+1)\frac{\pi}{2}\right)}{(2n+1)^3\pi^3} = \frac{16(-1)^n}{(2n+1)^3\pi^3}$$

$$\therefore\ A_n(f) = \frac{\frac{16(-1)^n}{(2n+1)^3\pi^3}}{\frac{1}{2}} = \frac{32(-1)^n}{(2n+1)^3\pi^3} \qquad (n = 0, 1, 2, \ldots)$$

The Fourier series of $f(x) = 1-x^2$ w.r.t. $\Phi = \left\{\cos\left(\frac{(2n+1)\pi x}{2}\right)\right\}_{n=0}^{\infty}$ on $[0,1]$ is

$$\sum_{n=0}^{\infty} A_n(f)\varphi_n(x) = \boxed{\sum_{n=0}^{\infty}\frac{32(-1)^n}{(2n+1)^3\pi^3}\cos\left(\frac{(2n+1)\pi x}{2}\right)}.$$

14 pts. to here.

(b) The orthogonal set $\Phi = \left\{\cos\left(\frac{(2n+1)\pi x}{2}\right)\right\}_{n=0}^{\infty}$ is the complete set of eigenfunctions for the symmetric operator $T = -\frac{d^2}{dx^2}$ on the space $V = \{\varphi \in C^2[0,1] : \varphi'(0) = 0 = \varphi(1)\}$. (See problem 6 on this exam.) Observe that $f(x) = 1-x^2$, $f'(x) = -2x$, and $f''(x) = -2$ are continuous functions on the interval $0 \le x \le 1$. Furthermore $f'(0) = 0$ and $f(1) = 0$ so f satisfies the symmetric boundary conditions of the space V. By the uniform convergence theorem (Theorem 2), the Fourier series of f in part (a) of this problem converges uniformly to f on $[0,1]$:

15 pts. to here.

$$1 - x^2 = f(x) = \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n\cos((2n+1)\pi x/2)}{(2n+1)^3} \qquad (0 \le x \le 1).$$

Since uniform convergence on $[0,1]$ implies L^2 and pointwise convergence on $[0,1]$, it follows that the Fourier series of f converges to f in the L^2 and pointwise senses on $[0,1]$.

25 pts. to here.

#6.
$$\begin{cases} u_{tt} - u_{xx} \overset{①}{=} 0 & \text{if } 0<x<1,\ 0<t<\infty, \\ u_x(0,t) \overset{②}{=} 0 \overset{③}{=} u(1,t) & \text{if } t \geq 0, \\ u_t(x,0) \overset{④}{=} 0 \text{ and } u(x,0) \overset{⑤}{=} 1-x^2 & \text{if } 0 \leq x \leq 1. \end{cases}$$

We use separation of variables. We seek nontrivial solutions to the homogeneous
1 portion of the problem, ①-②-③-④, of the form $u(x,t) = X(x)T(t)$. Substituting in ① gives

2
$$X(x)T''(t) - X''(x)T(t) = 0.$$

Dividing by $X(x)T(t)$ and rearranging produces

$$\frac{T''(t)}{T(t)} - \frac{X''(x)}{X(x)} = 0$$

so

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} = \text{constant} = \lambda.$$

3 pts. to here

Thus we are led to the coupled system of ODEs:

5
$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ T''(t) + \lambda T(t) = 0. \end{cases}$$

Applying ② and ③ yields $X'(0)T(t) = 0 = X(1)T(t)$ for all $t \geq 0$. If $u(x,t) = X(x)T(t)$ is not identically zero, then we must have $X'(0) = 0$ and $X(1) = 0$. Similarly ④ and nontriviality of $u(x,t) = X(x)T(t)$ imply $T'(0) = 0$. Therefore we need to solve

$$\begin{cases} X''(x) + \lambda X(x) \overset{⑥}{=} 0, \quad X'(0) \overset{⑦}{=} 0 \overset{⑧}{=} X(1), \\ T''(t) + \lambda T(t) \overset{⑨}{=} 0, \quad T'(0) \overset{⑩}{=} 0. \end{cases}$$

8 pts. to here.

Since the operator $T = -\frac{d^2}{dx^2}$ on $V = \{ \varphi \in C^2[0,1] : \varphi'(0) = 0 = \varphi(1) \}$ is symmetric (i.e. $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all $f, g \in V$), it follows that the eigenvalues

10 pts. to here. of ⑥-⑦-⑧ are real numbers. Furthermore, if f is a real eigenfunction for $T=-\frac{d^2}{dx^2}$ in V on $[0,1]$ corresponding to the eigenvalue λ then

$$\lambda\langle f,f\rangle=\langle \lambda f,f\rangle=\langle Tf,f\rangle=\int_0^1 \overset{dv}{-f''(x)}\underbrace{f(x)}_{u}dx=\cancel{-f(x)f'(x)\Big|_0^1}^{\,0}-\int_0^1 -f'(x)f'(x)dx$$

so $\lambda=\dfrac{\langle f',f'\rangle}{\langle f,f\rangle}\geq 0$. This shows that all eigenvalues of ⑥-⑦-⑧

12 are nonnegative.

Case 1: $\lambda>0$ (say $\lambda=\alpha^2$ where $\alpha>0$): Then ⑥ becomes $X''(x)+\alpha^2X(x)=0$ and the general solution is $X(x)=c_1\cos(\alpha x)+c_2\sin(\alpha x)$ with derivative $X'(x)=-\alpha c_1\sin(\alpha x)+\alpha c_2\cos(\alpha x)$. Applying ⑦ we have $0=X'(0)=-\alpha c_1\cdot 0+\alpha c_2\cdot 1$ so $c_2=0$. Applying ⑧ gives $0=X(1)=c_1\cos(\alpha)+\cancel{c_2}^{\,0}\sin(\alpha)=c_1\cos(\alpha)$. Clearly $c_1\neq 0$, for otherwise $X(x)=0$ for all $0\leq x\leq 1$, contradicting nontriviality of $u(x,t)=X(x)T(t)$.

14 Therefore $\cos(\alpha)=0$ is the eigenvalue condition in this case, so $\alpha=\alpha_n=(2n+1)\frac{\pi}{2}$

16 for some integer $n\geq 0$. Hence $\lambda_n=\alpha_n^2=\left((2n+1)\frac{\pi}{2}\right)^2$ and $X_n(x)=\cos(\alpha_n x)=$

18 $\cos\left(\frac{(2n+1)\pi x}{2}\right)$ $(n=0,1,2,\ldots)$ are the positive eigenvalues and corresponding eigenfunctions, respectively.

Case 2: $\lambda=0$: Then ⑥ becomes $X''(x)=0$ and the general solution is $X(x)=c_1x+c_2$ with derivative $X'(x)=c_1$. Applying ⑦, $0=X'(0)=c_1$ and applying ⑧, $0=X(1)=\cancel{c_1}^{\,0}+c_2=c_2$. Thus we only have trivial solutions

19 pts. to here. to ⑥-⑦-⑧ when $\lambda=0$; i.e. 0 is not an eigenvalue.

When $\lambda_n=\left(\frac{(2n+1)\pi}{2}\right)^2$ for some integer $n\geq 0$, ⑨ becomes $T_n''(t)+\left(\frac{(2n+1)\pi}{2}\right)^2T_n(t)=0$ with general solution $T_n(t)=c_1\cos\left(\frac{(2n+1)\pi t}{2}\right)+c_2\sin\left(\frac{(2n+1)\pi t}{2}\right)$ and derivative

$T_n'(t) = -\frac{(2n+1)\pi}{2} c_1 \sin\left(\frac{(2n+1)\pi t}{2}\right) + \frac{(2n+1)\pi}{2} c_2 \cos\left(\frac{(2n+1)\pi t}{2}\right)$. Applying (10) leads to

20 $0 = T_n'(0) = c_1 \cdot 0 + \frac{(2n+1)\pi}{2} c_2 \cdot 1$ and hence $c_2 = 0$. Thus $T_n(t) = \cos\left(\frac{(2n+1)\pi t}{2}\right)$,

pts. to here,

up to a constant factor. Hence

21 $$u_n(x,t) = \cos\left(\frac{(2n+1)\pi x}{2}\right)\cos\left(\frac{(2n+1)\pi t}{2}\right)$$

solves (1)-(2)-(3)-(4) for any integer $n \geq 0$. By the superposition principle,

22 $$u(x,t) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{(2n+1)\pi x}{2}\right)\cos\left(\frac{(2n+1)\pi t}{2}\right)$$

is a "formal" solution to (1)-(2)-(3)-(4) for "any" choice of constants $c_0, c_1, c_2, \ldots$

In order to satisfy (5),

$$1 - x^2 = u(x,0) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{(2n+1)\pi x}{2}\right)$$

for all $0 \leq x \leq 1$, it suffices by problem 5 of this exam to choose

23 $$c_n = A_n(f) = \frac{32(-1)^n}{(2n+1)^3\pi^3}$$

for $n = 0, 1, 2, \ldots$ Thus,

25 pts. to here $$\boxed{u(x,t) = \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n \cos((2n+1)\pi x/2)\cos((2n+1)\pi t/2)}{(2n+1)^3}}$$

solves (1)-(2)-(3)-(4)-(5). (Energy considerations can be used to show this is the only solution.)

#7.
$$\begin{cases} \nabla^2 u \overset{①}{=} 1 & \text{if } 1<r<3,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi, \\ u(1;\theta,\varphi) \overset{②}{=} 0 & \text{if } 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi, \\ u_r(3;\theta,\varphi) \overset{③}{=} 0 & \text{if } 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi. \end{cases}$$

Because the region, the PDE, and the BCs are invariant under rotations about the origin in 3-space, we expect the solution to be independent of the spherical coordinate angles θ and φ. That is, we expect u to be a radial function: $u = u(r;\theta,\varphi) = u(r)$. Hence $\frac{\partial u}{\partial \theta} = 0 = \frac{\partial u}{\partial \varphi}$ so ①,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2} = 1,$$

reduces to the ODE

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 1.$$

Multiplying by r^2 and integrating yields

$$r^2\frac{du}{dr} \overset{④}{=} \frac{r^3}{3} + c_1.$$

Dividing by r^2 and integrating gives

$$u \overset{⑤}{=} \frac{r^2}{6} - \frac{c_1}{r} + c_2.$$

From ③ we obtain $\frac{du}{dr}(3) = 0$ so substituting in ④ leads to

$$0 = \frac{3^3}{3} + c_1 \quad\Rightarrow\quad c_1 = -9.$$

Therefore
$$u(r) = \frac{r^2}{6} + \frac{9}{r} + c_2.$$

Applying ② we have $u(1) = 0$ so

$$0 = \frac{1^2}{6} + \frac{9}{1} + c_2 \quad\Rightarrow\quad c_2 = -\frac{55}{6}.$$

Consequently,
$$\boxed{u(r;\theta,\varphi) = \frac{r^2}{6} + \frac{9}{r} - \frac{55}{6}}$$

solves ①-②-③. (Energy considerations show that this solution is unique.)

#8. $$\begin{cases} u_t-(u_{xx}+u_{yy}) \overset{1}{=} 0 & \text{if } 0<x<\pi,\ 0<y<\pi,\ 0<t<\infty, \\ u_x(0,y,t) \overset{2}{=} 0 \overset{3}{=} u_x(\pi,y,t) & \text{if } 0\le y\le\pi,\ 0\le t<\infty, \\ u_y(x,0,t) \overset{4}{=} 0 \overset{5}{=} u_y(x,\pi,t) & \text{if } 0\le x\le\pi,\ 0\le t<\infty, \\ u(x,y,0) \overset{6}{=} \cos^2(x)\cos^2(y) & \text{if } 0\le x\le \pi,\ 0\le y\le\pi. \end{cases}$$

We use separation of variables. We seek nontrivial solutions to the homogeneous portion of the problem, 1-2-3-4-5, of the form $u(x,y,t)=X(x)Y(y)T(t)$. Substituting in 1 gives

1

$$X(x)Y(y)T'(t) - X''(x)Y(y)T(t) - X(x)Y''(y)T(t) = 0$$

2

and dividing by $X(x)Y(y)T(t)$ and rearranging yields

$$\frac{T'(t)}{T(t)} - \frac{X''(x)}{X(x)} - \frac{Y''(y)}{Y(y)} = 0$$

$$\frac{T'(t)}{T(t)} - \frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \text{constant} = -\mu$$

3

and then

$$\frac{T'(t)}{T(t)} + \mu = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda.$$

4 pts. to here

Therefore, we have the coupled system of ODEs:

$$\begin{cases} X''(x)+\lambda X(x) = 0, \\ Y''(y)+\mu Y(y) = 0, \\ T'(t) + (\lambda+\mu)T(t) = 0. \end{cases}$$

7 pts. to here.

Substituting in 2 and 3 we have

$$X'(0)Y(y)T(t) = 0 \quad \text{if } 0\le y\le\pi,\ 0\le t<\infty,$$
$$X'(\pi)Y(y)T(t) = 0 \quad \text{if } \quad " \quad , \quad " \quad .$$

In order that $u(x,y,t)=X(x)Y(y)T(t)$ is not the zero function, we must have $X'(0)=0=X'(\pi)$. Similarly 4 and 5 yield $Y'(0)=0=Y'(\pi)$. Thus, we obtain

$$\begin{cases} X''(x)+\lambda X(x) \overset{7}{=} 0, \quad X'(0) \overset{8}{=} 0 \overset{9}{=} X'(\pi), \\ Y''(y)+\mu Y(y) \overset{10}{=} 0, \quad Y'(0) \overset{11}{=} 0 \overset{12}{=} Y'(\pi), \\ T'(t) + (\lambda+\mu)T(t) \overset{13}{=} 0. \end{cases}$$

11 pts. to here.

By techniques from Sec. 5.2, we know the eigenvalues and eigenfunctions for ⑦-⑧-⑨ are

12 pts. to here. $$\lambda_\ell = \ell^2 \quad \text{and} \quad X_\ell(x) = \cos(\ell x), \qquad (\ell = 0, 1, 2, \ldots),$$

respectively. Similarly

13 $$\mu_m = m^2 \quad \text{and} \quad Y_m(y) = \cos(my), \qquad (m = 0, 1, 2, \ldots),$$

are the eigenvalues and eigenfunctions for ⑩-⑪-⑫. Substituting $\lambda = \lambda_\ell = \ell^2$ and $\mu = \mu_m = m^2$ into ⑬, it becomes

$$T'_{\ell,m}(t) + (\lambda_\ell + \mu_m) T_{\ell,m}(t) = 0$$

$$T'_{\ell,m}(t) + (\ell^2 + m^2) T_{\ell,m}(t) = 0$$

14 so $T_{\ell,m}(t) = e^{-(\ell^2+m^2)t}$, up to a constant factor. Hence

15 $$u_{\ell,m}(x,y,t) = X_\ell(x) Y_m(y) T_{\ell,m}(t) = \cos(\ell x)\cos(my)\, e^{-(\ell^2+m^2)t}$$

solves ①-②-③-④-⑤ for all integers $\ell \geq 0$ and $m \geq 0$. By the superposition principle,

16 $$u(x,y,t) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} c_{\ell,m} \cos(\ell x)\cos(my)\, e^{-(\ell^2+m^2)t}$$

is a "formal" solution to ①-②-③-④-⑤ for "every" choice of constants $c_{\ell,m}$. Using the identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ and applying the initial condition ⑥ yields

17 $$\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\right)\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2y)\right) = \cos^2(x)\cos^2(y) = u(x,y,0) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} c_{\ell,m} \cos(\ell x)\cos(my)$$

for all $0 \leq x \leq \pi$ and all $0 \leq y \leq \pi$. That is,

$$\frac{1}{4} + \frac{1}{4}\cos(2x) + \frac{1}{4}\cos(2y) + \frac{1}{4}\cos(2x)\cos(2y) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} c_{\ell,m} \cos(\ell x)\cos(my).$$

21 Therefore we should choose $c_{0,0} = \frac{1}{4}$, $c_{2,0} = \frac{1}{4}$, $c_{0,2} = \frac{1}{4}$, $c_{2,2} = \frac{1}{4}$ and all other $c_{\ell,m} = 0$.

Therefore

25 pts. to here. $$\boxed{u(x,y,t) = \frac{1}{4} + \frac{1}{4}\cos(2x)e^{-4t} + \frac{1}{4}\cos(2y)e^{-4t} + \frac{1}{4}\cos(2x)\cos(2y)e^{-8t}}$$

solves ①-②-③-④-⑤-⑥.

Math 325

Final Exam

Summer 2014

$n = 14$

$\mu = 169.2$

$\sigma = 18.1$

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
174 - 200	A	A	6
146 - 173	B	B	6
120 - 145	C	B	2
100 - 119	C	C	0
0 - 99	F	D	0