Math 5325 Exam I Summer 2015 Name: Dr. Grow

1.(20 pts.) Find the general solution of the first order partial differential equation

$$e^x u_x + xyu_y = 0.$$

2 pts. to here The characteristic curves of $a(x,y)u_x + b(x,y)u_y = 0$ satisfy $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$.

In our case this becomes

4 pts. to here
$$\frac{dy}{dx} = \frac{xy}{e^x}.$$

Separating variables, we have

6 pts. to here
$$\frac{dy}{y} = xe^{-x}dx$$

11 pts. to here so
$$\ln|y| = \int \frac{dy}{y} = \int \underbrace{x}_{u}\underbrace{e^{-x}dx}_{dv} \overset{\text{integrate by parts}}{=} \underbrace{-xe^{-x}}_{uv} - \int \underbrace{-e^{-x}}_{v}\underbrace{dx}_{du} = -xe^{-x} - e^{-x} + c.$$

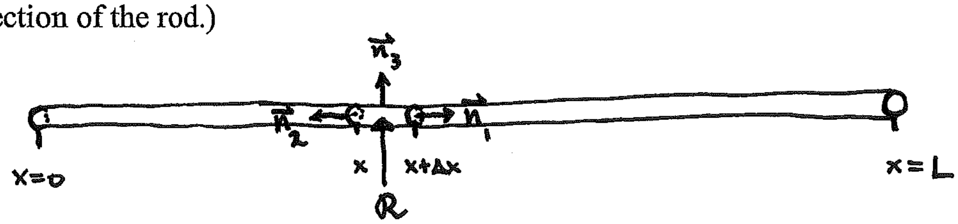
Therefore the characteristic curves have the form $\ln|y| + (x+1)e^{-x} = c$ or equivalently $y = A\exp(-(x+1)e^{-x})$ where $A = \pm e^c$ is an arbitrary constant.

Along a characteristic curve, the solution of $e^x u_x + xyu_y = 0$ is constant:

$$u(x, y(x)) = u(-1, y(-1)) = u(-1, A) = f(A).$$

20 pts. to here. Consequently, $\boxed{u(x,y) = f\left(y\exp((x+1)e^{-x})\right)}$ is the general solution of the PDE where f is an arbitrary C^1-function of a single real variable.

2.(20 pts.) On the lateral sides of a long thin rod, heat exchange takes place (obeying Newton's law of cooling – the heat flux is proportional to the temperature difference) with a medium of constant temperature T_0. Within the rod, heat exchange takes place according to Fourier's law – the velocity of the heat flow is proportional to the temperature gradient and is in the direction of decreasing temperature. Derive the equation satisfied by the temperature $u(x,t)$ at position x and time t within the rod. (You may assume the rod is so thin that you may neglect its variation in a cross section of the rod.)



Consider the material occupying the region R of the rod between positions x and $x+\Delta x$. We consider the rod sufficiently thin that we may safely neglect the variation of the temperature u over any cross section of the rod. Denote by $H(t)$ the heat energy contained in R. Then

$$(1) \qquad -\frac{dH}{dt} = \text{total flux of heat through } \partial R\text{, the boundary of } R$$

$$= \iint_{\partial R} \vec{v}\cdot\vec{n}\, d\sigma$$

$$= \underbrace{\iint \overbrace{-Ku_x(x+\Delta x,t)}^{\vec{v}\cdot\vec{n}_1}\, d\sigma}_{\text{Right End}} + \underbrace{\iint \overbrace{Ku_x(x,t)}^{\vec{v}\cdot\vec{n}_2}\, d\sigma}_{\text{Left End}} + \underbrace{\iint \overbrace{\mu(u(z,t)-T_0)}^{\vec{v}\cdot\vec{n}_3}\, d\sigma}_{\text{Lateral Wall}}$$

3 pts. to here.

where

$$\vec{v} = \text{velocity of the heat flow} = \begin{cases} -K\nabla u & \text{for conductive heat flow on ends,} \\ \mu(u-T_0)\vec{n}_3 & \text{for convective heat flow on sides,} \end{cases}$$

5 pts. to here.

and K and μ are positive constants. The element of area on the ends is $d\sigma = r\,dr\,d\theta$ (I'm assuming circular cross sections of radius R and I'm using polar coordinates) and on the lateral wall $d\sigma = R\,d\theta\,dz$. Performing the integrations in the last line of (1) gives

$$(2) \qquad -\frac{dH}{dt} = \int_0^R \int_0^{2\pi} -Ku_x(x+\Delta x,t)\, r\, d\theta\, dr + \int_0^R \int_0^{2\pi} Ku_x(x,t)\, r\, d\theta\, dr + \int_x^{x+\Delta x} \int_0^{2\pi} \mu\big(u(z,t)-T_0\big) R\, d\theta\, dz$$

$$= \frac{R^2}{2}(2\pi)\big(-Ku_x(x+\Delta x,t)\big) + \frac{R^2}{2}(2\pi)\big(Ku_x(x,t)\big) + 2\pi R \int_x^{x+\Delta x} \mu\big(u(z,t)-T_0\big)\, dz$$

7 pts. to here.

$$= -KA\big(u_x(x+\Delta x,t) - u_x(x,t)\big) + P\int_x^{x+\Delta x} \mu\big(u(z,t)-T_0\big)\, dz .$$

(Here A and P are the area and perimeter, respectively, of the (uniform) cross sections.)

On the other hand, the heat in $\mathcal{R}$ is given by

9 pts. to here.

$$(3) \qquad H(t) = \iiint_{\mathcal{R}} E(\vec{x}, u(\vec{x},t))\, d\vec{x}$$

where $E(\vec{x}, u)$ is the heat energy density function for the material occupying $\mathcal{R}$. Differentiating (3) and assuming smoothness of E gives

11 pts. to here.

$$(4) \qquad \frac{dH}{dt} = \frac{d}{dt}\left(\iiint_{\mathcal{R}} E(\vec{x},u(\vec{x},t))\, d\vec{x}\right) = \iiint_{\mathcal{R}} \frac{\partial}{\partial t} E(\vec{x},u(\vec{x},t))\, d\vec{x} = \iiint_{\mathcal{R}} \frac{\partial E}{\partial u}\frac{\partial u}{\partial t}\, d\vec{x} .$$

13 pts. to here. For normal temperature ranges and typical materials, $\frac{\partial E}{\partial u} \cong$ constant $= c\rho$ where c is the specific heat of the material and ρ its density. Substituting in (4) and, using cylindrical coordinates r, θ, z so the volume element is $d\vec{x} = r\,dr\,d\theta\,dz$, we have

$$(5) \qquad \frac{dH}{dt} = \int_x^{x+\Delta x} \int_0^{2\pi} \int_0^R c\rho \frac{\partial u}{\partial t}(z,t)\, r\, dr\, d\theta\, dz = c\rho\left(\frac{R^2}{2}\right)(2\pi) \int_x^{x+\Delta x} \frac{\partial u}{\partial t}(z,t)\, dz$$

15 pts. to here.

$$= c\rho A \int_x^{x+\Delta x} \frac{\partial u}{\partial t}(z,t)\, dz .$$

Comparing (2) and (5) we have

17 pts. to here.

$$(6) \qquad c\rho A \int_x^{x+\Delta x} u_t(z,t)\, dz = KA\big(u_x(x+\Delta x,t) - u_x(x,t)\big) - P\int_x^{x+\Delta x} \mu\big(u(z,t)-T_0\big)\, dz .$$

Dividing through (b) by Δx and letting $\Delta x \to 0$ yields

$$c\rho A \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} u_t(z,t)\,dz = KA \lim_{\Delta x \to 0} \frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} - P\mu \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} (u(z,t) - T_0)\,dz$$

or equivalently

$$c\rho A\, u_t(x,t) = KA\, u_{xx}(x,t) - P\mu\left(u(x,t) - T_0\right).$$

This can be expressed alternatively as

20 pts. to here.

$$\boxed{u_t(x,t) = \frac{K}{c\rho} u_{xx}(x,t) - \frac{\mu P}{c\rho A}\left(u(x,t) - T_0\right)}.$$

(Here K is the conductivity of the material occupying $\mathcal{R}$, μ is the surface conductance, c is the specific heat, ρ is the density, A is the cross sectional area, P is the perimeter of a cross section, and T_0 is the ambient temperature of the medium surrounding the rod.)

3.(40 pts.) Classify the type (parabolic, hyperbolic, elliptic) of each of the following second order linear partial differential equations and, whenever possible, find the general solution in the xy – plane.

(a) $u_{xx} - 4u_{xy} + 3u_{yy} = \sin(x+y)$

(b) $u_{xx} + 2u_{xy} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$

(c) $u_{xx} + u_{yy} - 2u_{xy} + u_x - u_y = 0$

2 pts. (a) $B^2 - 4AC = (-4)^2 - 4(1)(3) = 4 > 0$. The PDE in (a) is $\boxed{\text{hyperbolic}}$.

2 pts. (b) $B^2 - 4AC = (2)^2 - 4(1)(3) = -8 < 0$. The PDE in (b) is $\boxed{\text{elliptic}}$.

2 pts. (c) $B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$. The PDE in (c) is $\boxed{\text{parabolic}}$.

18 pts.) Solution of the PDE in (a): We write a PDE equivalent to that in (a):

$$\left(\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x \partial y} + 3\frac{\partial^2}{\partial y^2}\right)u = \sin(x+y)$$

1 pt. to here.

$$(*) \qquad \left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = \sin(x+y).$$

The suggested change of variables in a hyperbolic PDE is

5 pts. to here.

$$\xi = -(\beta x - \alpha y) = -[(-3)x - (1)y] = 3x + y$$
$$\eta = -(\delta x - \gamma y) = -[(-1)x - (1)y] = x + y.$$

If v is a smooth function of two variables x and y and we perform the above change of variables, then the chain rule gives

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = 3\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$$
$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}.$$

That is, as operators, $\frac{\partial}{\partial x} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$. Substituting these expressions in (*) yields

$$\left(\cancel{3\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta} - 3\left(\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right)\right)\left(3\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}} - \left(\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}}\right)\right)u = \sin(\eta)$$

$$\left(-2\frac{\partial}{\partial\eta}\right)\left(2\frac{\partial}{\partial\xi}\right)u = \sin(\eta)$$

10 pts. to here

$$\frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial\xi}\right) = -\frac{1}{4}\sin(\eta)$$

Integrating with respect to η holding ξ fixed gives

12 pts. to here.

$$\frac{\partial u}{\partial\xi} = \int -\frac{1}{4}\sin(\eta)\,d\eta = \frac{1}{4}\cos(\eta) + c(\xi).$$

Integrating again, but this time with respect to ξ, holding η fixed, produces

14 pts. to here.

$$u = \int\left[\frac{1}{4}\cos(\eta) + c(\xi)\right]d\xi = \frac{\xi}{4}\cos(\eta) + \int c(\xi)\,d\xi.$$

Therefore

$$u = \frac{1}{4}\xi\cos(\eta) + f(\xi) + g(\eta)$$

where f and g are arbitrary C^2-functions of a single real variable. Hence

18 pts. to here

$$\boxed{u(x,y) = \frac{1}{4}(3x+y)\cos(x+y) + f(3x+y) + g(x+y)}$$

is the general solution of (a) in the xy-plane.

(18 pts.) <u>Solution of the PDE in (c):</u> We write a PDE equivalent to that in (c):

$$\left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y^2}\right)u + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = 0$$

1 pt. to here.

$$(**) \quad \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = 0.$$

The suggested change of variables in a parabolic PDE is

$$\xi = -(\beta x - \alpha y) = -[(-1)x - (1)y] = x+y$$

5 pts. to here.

$$\eta = \alpha x + \beta y = (1)x + (-1)y = x-y.$$

Using the chain rule as in part (a), we find that as operators

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}.$$

Substituting these expressions in (**) leads to

$$\left[\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta} - \left(\cancel{\frac{\partial}{\partial \xi}} - \frac{\partial}{\partial \eta}\right)\right]\left[\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta} - \left(\cancel{\frac{\partial}{\partial \xi}} - \frac{\partial}{\partial \eta}\right)\right]u + \left[\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta} - \left(\cancel{\frac{\partial}{\partial \xi}} - \frac{\partial}{\partial \eta}\right)\right]u = 0$$

10 pts. to here.

$$(***) \qquad 4\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial u}{\partial \eta} = 0.$$

This last equation is really an ODE in the independent variable η since there are no derivatives with respect to ξ. We seek exponential solutions, $u = e^{r\eta}$, where r is a constant. Then $\frac{\partial u}{\partial \eta} = re^{r\eta}$ and $\frac{\partial^2 u}{\partial \eta^2} = r^2 e^{r\eta}$ so substituting in (***) gives $4r^2e^{r\eta} + 2re^{r\eta} = 0 \Rightarrow 2(2r+1)re^{r\eta} = 0$ so $r = -\frac{1}{2}$ or $r = 0$.

Therefore $u_1 = e^{0\eta} = 1$ and $u_2 = e^{-\frac{1}{2}\eta}$ are (linearly independent) solutions to (***).

The general solution of (***) is

16 pts. to here.

$$u = c_1(\xi)\cdot 1 + c_2(\xi)e^{-\frac{1}{2}\eta}.$$

The general solution of the PDE in (c) consequently is

18 pts. to here.

$$\boxed{u(x,y) = f(x+y) + g(x+y)e^{-\frac{1}{2}(x-y)}}$$

where f and g are arbitrary C^2-functions of a single real variable.

4.(20 pts.) A homogenous body occupying the solid region D is completely insulated. The initial temperature at a general point (x, y, z) in D is given by a prescribed continuous function $\varphi = \varphi(x, y, z)$. Derive an expression (in terms of φ) for the constant steady-state temperature reached by the body after a long time.

The (transient) temperature $u = u(\vec{x}, t)$ of the material at position $\vec{x}$ in D and time $t \geq 0$ satisfies the initial/boundary value problem

$$\begin{aligned} u_t(\vec{x},t) &= K \nabla^2 u(\vec{x},t) && \text{if } \vec{x} \in D \text{ and } t \geq 0, \\ u(\vec{x},0) &= \varphi(\vec{x}) && \text{if } \vec{x} \in D, \\ \vec{n} \cdot \nabla u(\vec{x},t) &= 0 && \text{if } \vec{x} \in \partial D \text{ and } t \geq 0. \end{aligned}$$

Since the system is closed (i.e. the boundary of D is insulated and there are no sources or sinks of heat in D), it can be shown that the total

5 pts. to here.

heat energy of the material occupying D is constant: $H(0) = H(t)$ for all $t \geq 0$. *

We assume that the transient temperature tends toward a constant steady-state temperature: $\lim_{t \to \infty} u(\vec{x},t) = \text{constant} = U$. Therefore

10 pts. to here.

$$c\rho \iiint_D \varphi(\vec{x})\,d\vec{x} = \iiint_D c\rho u(\vec{x},0)\,d\vec{x} = H(0) = \lim_{t\to\infty} H(t) = \lim_{t\to\infty} \iiint_D c\rho\, u(\vec{x},t)\,d\vec{x}$$

so

15 pts. to here.

$$c\rho \iiint_D \varphi(\vec{x})\,d\vec{x} = c\rho \iiint_D \lim_{t\to\infty} u(\vec{x},t)\,d\vec{x} = c\rho \iiint_D U\,d\vec{x} = c\rho U \,\mathrm{vol}(D).$$

Cancelling $c\rho$ and solving for U gives

20 pts. to here.

$$\boxed{U = \frac{1}{\mathrm{vol}(D)} \iiint_D \varphi(\vec{x})\,d\vec{x}} \left(= \text{average value of } \varphi \text{ on } D\right).$$

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* The total energy of the material occupying D at time t is actually given

by

$$\tilde{H}(t) = \iiint_D E(\vec{x}, u(\vec{x},t))\,d\vec{x}$$

where $E = E(\vec{x}, u)$ is the energy density function at position $\vec{x}$ in D and temperature u. Since $\frac{\partial E}{\partial u} \cong c\rho$ for large temperature ranges, it is convenient to use

$$H(t) = \iiint_D c\rho u(\vec{x},t)\,d\vec{x}$$

to approximate $\tilde{H}(t)$. Note that

$$\frac{dH}{dt} = \frac{d}{dt}\left(\iiint_D c\rho u(\vec{x},t)\,d\vec{x}\right) = \iiint_D c\rho u_t(\vec{x},t)\,d\vec{x} = c\rho K \iiint_D \nabla^2 u(x,t)\,d\vec{x}.$$

Therefore, the divergence theorem and the homogeneous Neumann B.C. yields

$$\frac{dH}{dt} = c\rho K \iiint_D \nabla\cdot(\nabla u) = c\rho K \iint_{\partial D} \vec{n}\cdot\nabla u(x,t)\,d\sigma = c\rho K \iint_{\partial D} 0\,d\sigma = 0.$$

That is, $H(t) = H(0)$ for all $t \geq 0$.

Distribution of Scores on Exam I

Range	Graduate Letter Grade	Undegraduate Letter Grade	Frequency
87 - 100	A	A	1
73 - 86	B	B	3
60 - 72	C	B	4
50 - 59	C	C	3
0 - 49	F	D	5

number of exams: 16

mean : 58.3

median : 59.5

standard deviation: 20.9