Mathematics 5325 Exam II Summer 2015 Name: Dr. Grow

1.(25 pts.) Solve $u_t - u_{xx} = 0$ in the upper half-plane $-\infty < x < \infty$, $0 < t < \infty$, subject to the initial condition $u(x,0) = x^2$ if $-\infty < x < \infty$. You may find the following identities useful:

$$2\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}\,,\quad \int_{-\infty}^{\infty} pe^{-p^2}\,dp = 0\,.$$

(*) $u(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\varphi(y)\,dy$ solves $u_t - ku_{xx} = 0$ in the upper half-plane

2 pts. to here.

subject to the initial condition $u(x,0) = \varphi(x)$ if $-\infty < x < \infty$; here φ is a bounded, continuous function on $(-\infty,\infty)$. In our case $k=1$ and $\varphi(x) = x^2$ is continuous but not bounded on $(-\infty,\infty)$. Therefore, after we apply (*), we will need to check our answer.

8 pts. to here

$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^2\,dy$. Let $p = \frac{y-x}{\sqrt{4t}}$. Then $dp = \frac{dy}{\sqrt{4t}}$, if $y \to +\infty$ then $p \to \infty$, and if $y \to -\infty$ then $p \to -\infty$. Therefore

17 pts. to here

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-p^2}(x + p\sqrt{4t})^2\sqrt{4t}\,dp = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\left(x^2 + 2xp\sqrt{4t} + 4tp^2\right)dp$$

20 pts. to here

$$= \frac{x^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\,dp + \frac{2x\sqrt{4t}}{\sqrt{\pi}}\int_{-\infty}^{\infty} pe^{-p^2}\,dp + \frac{4t}{\sqrt{\pi}}\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp.$$

23 pts. to here.

Using the given identities yields $u(x,t) = x^2 + 0 + 4t\cdot\frac{1}{2} = \boxed{x^2 + 2t}$.

25 pts. to here.

Check: $u_t - u_{xx} = 2 - 2 \overset{\checkmark}{=} 0$

$u(x,0) \overset{\checkmark}{=} x^2$

2.(25 pts.) For a solution to the wave equation $\boxed{\overset{(*)}{\rho u_{tt} - Tu_{xx} = 0}}$, the energy density is defined by

$$e = \frac{1}{2}\rho u_t^2 + \frac{1}{2}Tu_x^2$$

and the momentum density as

$$p = u_t u_x .$$

(a) Show that $\dfrac{\partial e}{\partial t} = T\dfrac{\partial p}{\partial x}$ and $\dfrac{\partial e}{\partial x} = \rho\dfrac{\partial p}{\partial t}$.

(b) If u is a C^3 function, show that e and p also satisfy the wave equation.

12 pts. (a) (2) $\dfrac{\partial e}{\partial t} = \dfrac{\partial}{\partial t}\left(\frac{1}{2}\rho u_t^2 + \frac{1}{2}Tu_x^2\right) = \rho u_t u_{tt} + T u_x u_{xt} \overset{(*)}{=} T u_t u_{xx} + T u_x u_{xt}$.

(2) $\dfrac{\partial p}{\partial x} = \dfrac{\partial}{\partial x}(u_t u_x) = u_{tx} u_x + u_t u_{xx}$. From these two identities and the

(2) standard assumption that $u = u(x,t)$ is C^2 (so $u_{xt} = u_{tx}$), it follows that $\boxed{\dfrac{\partial e}{\partial t} = T\dfrac{\partial p}{\partial x}}$.

(2) $\dfrac{\partial e}{\partial x} = \dfrac{\partial}{\partial x}\left(\frac{1}{2}\rho u_t^2 + \frac{1}{2}Tu_x^2\right) = \rho u_t u_{tx} + T u_x u_{xx} \overset{(*)}{=} \rho u_t u_{tx} + \rho u_x u_{tt}$.

(2) $\dfrac{\partial p}{\partial t} = \dfrac{\partial}{\partial t}(u_t u_x) = u_{tt} u_x + u_t u_{xt}$. From these two identities and the

(2) fact that $u_{tx} = u_{xt}$, it follows that $\boxed{\dfrac{\partial e}{\partial x} = \rho\dfrac{\partial p}{\partial t}}$.

13 pts. (b) Let u be a C^3-solution to (*). Then $\dfrac{\partial^2 p}{\partial t\partial x} = \dfrac{\partial^2 p}{\partial x \partial t}$ (1), so by part (a)

(6) $$\boxed{\rho\frac{\partial^2 e}{\partial t^2} - T\frac{\partial^2 e}{\partial x^2}} = \rho\frac{\partial}{\partial t}\left(\frac{\partial e}{\partial t}\right) - T\frac{\partial}{\partial x}\left(\frac{\partial e}{\partial x}\right) = \rho T\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial x}\right) - T\rho\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial t}\right) = \boxed{0.}$$

Also since u is a C^3-solution to (*), $\dfrac{\partial^2 e}{\partial t\partial x} = \dfrac{\partial^2 e}{\partial x\partial t}$. Hence by part (a),

(6) $$\boxed{\rho\frac{\partial^2 p}{\partial t^2} - T\frac{\partial^2 p}{\partial x^2}} = \rho\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial t}\right) - T\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\partial e}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial e}{\partial t}\right) = \boxed{0.}$$

That is, e and p are solutions to the wave equation (*).

3.(25 pts.) In this problem, you may find useful the fact that the Laplacian of u in spherical coordinates is given by

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial u}{\partial \varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}.$$

A spherical wave is a solution of the three-dimensional wave equation of the form $u = u(r,t)$, independent of the spherical coordinates φ and θ; here r denotes the distance from the origin.

(a) Show that a spherical wave satisfies the partial differential equation $u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$.

(b) Make the change of variables $v(r,t) = ru(r,t)$ in the partial differential equation in (a) and show that v satisfies the partial differential equation $v_{tt} - c^2 v_{rr} = 0$.

(c) Find the spherical wave $u = u(r,t)$ satisfying the initial conditions $u(r,0) = r^2$ and $u_t(r,0) = 3rc$ for all $r > 0$. (Hint: Use part (b).)

5 pts. (a) Let $u = u(r,t)$ be a solution of the three-dimensional wave equation $u_{tt} - c^2\nabla^2 u = 0$. (1) Using the given expression for $\nabla^2 u$ in spherical coordinates and the fact that $\frac{\partial u}{\partial \varphi} = 0 = \frac{\partial u}{\partial \theta}$ (1) (1) for a spherical wave, we find that

$$u_{tt} = c^2\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)\right) = c^2\left(\frac{1}{r^2}\left(r^2\frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r}\right)\right) = c^2\left(u_{rr} + \frac{2}{r}u_r\right). \quad (2)$$

5 pts. (b) Let $u = u(r,t)$ be a solution of $u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$ and define $v(r,t) = ru(r,t)$. Then $v_r(r,t) = ru_r(r,t) + u(r,t)$ (1) and $v_{rr}(r,t) = ru_{rr}(r,t) + 2u_r(r,t)$. (1) Consequently,

$$\boxed{v_{tt} - c^2 v_{rr}} = ru_{tt} - c^2(ru_{rr} + 2u_r) = r\left[u_{tt} - c^2\left(u_{rr} + \frac{2}{r}u_r\right)\right] = \boxed{0,}$$ at least (3)

for $r > 0$.

15 pts. (c) Let $u = u(r,t)$ be a spherical wave satisfying the initial conditions $u(r,0) = r^2$ and $u_t(r,0) = 3rc$ for $r > 0$. Then $u = u(r,t)$ satisfies the partial differential equation in part (a), and applying part (b), $v(r,t) = ru(r,t)$ satisfies

(5) (**) $$v_{tt} = c^2 v_{rr} \quad \text{subject to } v(r,0) = ru(r,0) = r^3 \text{ and } v_t(r,0) = ru_t(r,0) = 3r^2c.$$

By d'Alembert's formula the solution to (**) in the domain $-\infty < r < \infty$, $0 < t < \infty$, is given by

$$v(r,t) = \frac{1}{2}\left[\varphi(r-ct) + \varphi(r+ct)\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} \psi(s)\,ds$$ (4)

$$= \frac{1}{2}\left[(r-ct)^3 + (r+ct)^3\right] + \frac{1}{2\cancel{c}}\int_{r-ct}^{r+ct} 3s^2\cancel{c}\,ds$$

$$= \frac{1}{2}\left[\cancel{(r-ct)^3} + (r+ct)^3\right] + \frac{1}{2}\left[(r+ct)^3 - \cancel{(r-ct)^3}\right]$$ (4)

$$= (r+ct)^3.$$

Therefore $\boxed{u(r,t) =} \frac{1}{r} v(r,t) = \boxed{\frac{1}{r}(r+ct)^3}$ is a candidate for a solution (2)

to $u_{tt} = c^2(u_{rr} + \frac{2}{r}u_r)$ in $0<r<\infty,\ 0<t<\infty$, subject to the initial conditions

$u(r,0) = r^2$ and $u_t(r,0) = 3rc$ for $r>0$.

<u>Check:</u> $$u_{tt} - c^2(u_{rr} + \frac{2}{r}u_r) = \frac{6}{r}(r+ct)c^2 - c^2\left(2 + \cancel{\frac{2c^3t^3}{r^3}} + 2\left(2 + \frac{3ct}{r} - \cancel{\frac{c^3t^3}{r^3}}\right)\right)$$

$$= c^2\left(6 + \frac{6ct}{r} - 2 - 4 - \frac{6ct}{r}\right)$$

$$\stackrel{\checkmark}{=} 0 \qquad (0<r<\infty,\ 0<t<\infty).$$

$$u(r,0) = \frac{1}{r}(r+0)^3 \stackrel{\checkmark}{=} r^2 \qquad (0<r<\infty).$$

$$u_t(r,t) = \frac{3}{r}(r+ct)^2 c \qquad (0<r<\infty,\ 0<t<\infty).$$

$$u_t(r,0) = \frac{3}{r}(r+0)^2 c \stackrel{\checkmark}{=} 3rc \qquad (0<r<\infty).$$

4.(25 pts.) (a) Complete the entries in the following table comparing properties of solutions to the wave and diffusion equations.
(b) Give reasons for the entries giving the behavior of waves and diffusions as $t \to \infty$.

(a) 14 pts.

Property	Solutions to Wave Equation	Solutions to Diffusion Equation
Speed of propagation?	speed $\le c$ so finite propagation speed.	Infinite speed of propagation.
Singularities for $t > 0$?	Yes and they are transported with speed c along characteristics.	No, singularities dissipate immediately for $t>0$.
Well-posed for $t > 0$?	Yes.	Yes, at least for bounded solutions.
Well-posed for $t < 0$?	Yes.	No, backward diffusion can lead to chaos.
Maximum principle?	No.	Yes, on bounded domains.
Behavior as $t \to \infty$?	Energy is constant over time so solutions do not decay as $t \to \infty$.	Solutions decay to zero as $t\to\infty$, at least for absolutely integrable initial distributions φ.
Information transported?	Yes, along characteristics.	No, information degrades gradually as time increases.

11 pts. (b) If u solves $\rho u_{tt} - Tu_{xx} = 0$ and satisfies decay conditions (a) and (b) in the

Waves (5)

conservation of energy theorem, then $E(t) = \int_{-\infty}^{\infty} \left[\frac{1}{2}\rho u_t^2(x,t) + \frac{1}{2}T u_x^2(x,t)\right]dx$ satisfies $\frac{dE}{dt} = 0$ and hence $E(t) = E(0)$ for $t \ge 0$. Therefore the solution $u = u(x,t)$ does not decay as $t \to \infty$ unless $u(x,0) = 0 = u_t(x,0)$ for all $-\infty < x < \infty$.

Diffusions (6)

If u solves $u_t - ku_{xx} = 0$ in the upper half-plane and $u(x,0) = \varphi(x)$ for $-\infty<x<\infty$ where φ is bounded, continuous, and absolutely integrable then

$$|u(x,t)| = \left|\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\varphi(y)\,dy\right| \le \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty}\left|e^{-\frac{(x-y)^2}{4kt}}\varphi(y)\right|dy \le \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty}|\varphi(y)|\,dy$$

$\le \frac{C}{\sqrt{t}}$ for the real number $C = \frac{1}{\sqrt{4k\pi}}\int_{-\infty}^{\infty}|\varphi(y)|\,dy$ and all $t>0$. Hence $|u(x,t)| \to 0$ (uniformly) for $-\infty<x<\infty$ as $t \to +\infty$.

Math 5325

Exam II

Summer 2015

number taking exam = 16

mean score = 69.6

median score = 71.5

standard deviation = 15.9

Distribution of scores:

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87 – 100	A	A	3
73 – 86	B	B	5
60 – 72	C	B	2
50 – 59	C	C	4
0 – 49	F	D	2