Mathematics 325 | Final Exam Fall 2012 | Name: Dr. Grow

1.(33 pts.) Classify the following second order linear partial differential equations as elliptic, parabolic, or hyperbolic and find the general solution whenever possible.

(a) $u_{xx} - 4u_{xy} + 5u_{yy} - 3u_y = 0$ (b) $u_{xx} + u_{yy} - 2u_{xy} + u_x - u_y = 0$

2.(33 pts.) A homogeneous solid material occupying the hollowed out cylinder

$$C = \{(x,y,z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 \le 4,\ 0 \le z \le 2\}$$

is completely insulated and its initial temperature at position (x,y,z) in C is $50/\sqrt{x^2+y^2}$.

(a) Write, without proof or derivation, the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) in C and time $t \ge 0$.

(b) Use the divergence theorem to help show that the heat

$$H(t) = \iiint_C c\rho u(x,y,z,t)\,dV$$

of the material in C at time t is a constant function of time. Here c and ρ denote the specific heat and mass density, respectively, of the material in C.

(c) Compute the steady-state temperature that the material in C reaches after a long time.

3.(33 pts.) (a) Under appropriate hypotheses on a function $g = g(x)$ defined on $-\infty < x < \infty$, show that

$$\mathfrak{F}(g(x-a))(\xi) = \hat{g}(\xi)e^{-i\xi a} \quad \text{and} \quad \mathfrak{F}\left(\int_{-\infty}^{x} g(s)\,ds\right)(\xi) = \frac{\hat{g}(\xi)}{i\xi}.$$

(b) Use Fourier transform methods to derive a formula for the solution to the nonhomogeneous wave equation $u_{tt} - c^2 u_{xx} = f(x,t)$ in $-\infty < x < \infty,\ -\infty < t < \infty$, subject to $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ if $-\infty < x < \infty$. You may find it useful to recall the identities $\cos(\theta) = (e^{i\theta} + e^{-i\theta})/2$ and $\sin(\theta) = (e^{i\theta} - e^{-i\theta})/2i$.

4.(33 pts.) Solve the beam equation $u_{tt} + u_{xxxx} = 0$ if $0 < x < 1,\ 0 < t < \infty$, subject to the boundary conditions $u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0$ if $t \ge 0$, and the initial conditions $u(x,0) = 2\sin(\pi x) - 3\sin(5\pi x)$ and $u_t(x,0) = 0$ if $0 \le x \le 1$. You may find useful the fact that the general solution to $y^{(4)} - \beta^4 y = 0$ is $y(x) = c_1\cosh(\beta x) + c_2\sinh(\beta x) + c_3\cos(\beta x) + c_4\sin(\beta x)$.

5.(34 pts.) (a) Find a solution to the heat equation $u_t - u_{xx} = 0$ on $0 < x < 1,\ 0 < t < \infty$, subject to $u_x(0,t) = 0 = u(1,t)$ if $t \ge 0$ and $u(x,0) = 1 - x^2$ if $0 \le x \le 1$. Be sure to show convergence of any Fourier series that you use to represent a function when solving this problem.

(b) Is the solution to the problem in part (a) unique? Justify your answer.

6.(34 pts.) (a) Find a solution to $\nabla^2 u = 0$ in the cube C: $0 < x < 1,\ 0 < y < 1,\ 0 < z < 1$, subject to the nonhomogeneous Dirichlet boundary condition $u(x,y,1) = \sin(\pi x)\sin^3(\pi y)$ for $0 \le x \le 1,\ 0 \le y \le 1$ and homogeneous Dirichlet conditions on the other five faces of C. You may find the identity $\sin^3(A) = (3\sin(A) - \sin(3A))/4$ useful.

(b) State the maximum/minimum principle for harmonic functions and use it to show that the problem in part (a) has only one solution.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$
A.	$\begin{cases} 1 & \text{if } -b<x<b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c<x<d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2+a^2} \quad (a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0<x\le b, \\ 2b-x & \text{if } b<x<2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x>0, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b<x<c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b<x<b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases} e^{iax} & \text{if } c<x<d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a>0)$	$\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a>0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert\ge a, \\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert<a. \end{cases}$

Convergence Theorems

Consider the eigenvalue problem

(1) $X''(x)+\lambda X(x)=0$ in $a<x<b$ with boundary conditions that make $T=-\dfrac{d^2}{dx^2}$ symmetric

and let $\Phi=\{X_1,X_2,X_3,...\}$ be a complete orthogonal set of eigenfunctions for (1). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$\sum_{n=1}^{\infty} A_n X_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle} \qquad (n=1,2,3,...).$$

Theorem 2. (Uniform Convergence) If

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and

(ii) f satisfies the given symmetric boundary conditions,

then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 – Convergence) If

$$\int_a^b |f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)

(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.

(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty} A_n X_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x.

#1. (a) $B^2 - 4AC = (-4)^2 - 4(1)(5) = -4 < 0$ $\boxed{\text{elliptic}}$ (general solution impossible) 3

6 pts. (b) $B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$ $\boxed{\text{parabolic}}$ 3

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27 pts. We find the general solution of (b) by first rewriting the equation as

$$\left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right)u + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = 0 \quad \text{or} \quad \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 u + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = 0. \quad 3$$

This suggests the change-of-coordinates:

$$\xi = -(\beta x - \alpha y) = -(-x-y) = x+y \quad 3$$
$$\eta = \alpha x + \beta y = x - y = x - y. \quad 3$$

The chain rule gives $\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \overset{1}{=} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} \overset{1}{=} \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$. Hence $\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) = 2\frac{\partial}{\partial \eta}$ 1, so the p.d.e. in (d) can be written as $4\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial u}{\partial \eta} \overset{2}{=} 0$. Writing $v = \frac{\partial u}{\partial \eta}$ this becomes

$4\frac{\partial v}{\partial \eta} + 2v = 0 \iff \frac{\partial v}{\partial \eta} + \frac{1}{2}v \overset{2}{=} 0$. An integrating factor is $\mu = e^{\int p(\eta)d\eta} = e^{\int \frac{1}{2}d\eta} \overset{2}{=} e^{\eta/2 + \cancel{c}^{\,0}}$. Then $e^{\frac{\eta}{2}}\cdot\frac{\partial v}{\partial \eta} + \frac{1}{2}e^{\frac{\eta}{2}}v = 0$ or $\frac{\partial}{\partial \eta}\left(e^{\frac{\eta}{2}}v\right) \overset{2}{=} 0$

so $e^{\frac{\eta}{2}}v \overset{1}{=} c_1(\xi)$. But then $\frac{\partial u}{\partial \eta} = v = c_1(\xi)e^{-\eta/2}$ so integrating with respect to η holding ξ fixed gives $u = \int c_1(\xi)e^{-\eta/2}d\eta = -2c_1(\xi)e^{-\eta/2} + c_2(\xi)$.

In other words $u \overset{3}{=} f(\xi)e^{-\eta/2} + g(\xi)$ where f and g are any C^2-functions of a single real variable. Thus

$$\boxed{u(x,y) \overset{2}{=} f(x+y)e^{\frac{1}{2}(y-x)} + g(x+y)}$$

is the general solution of (b).

2.(33 pts.) A homogeneous solid material occupying the hollowed out cylinder

$$C=\{(x,y,z)\in\mathbb{R}^3 : 1\le x^2+y^2\le 4,\ 0\le z\le 2\}$$

is completely insulated and its initial temperature at position (x,y,z) in C is $50/\sqrt{x^2+y^2}$.

(a) Write, without proof or derivation, the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) in C and time $t\ge 0$.

(b) Use the divergence theorem to help show that the heat energy

$$H(t)=\iiint_C c\rho u(x,y,z,t)\,dV$$

of the material in C at time t is a constant function of time. Here c and ρ denote the specific heat and mass density, respectively, of the material in C.

(c) Compute the steady-state temperature that the material in C reaches after a long time.

15 pts. (a)

$$\begin{cases} (1)\quad u_t - k\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right) \overset{(5)}{=} 0 & \text{if } (x,y,z) \text{ is in the interior of } C \text{ and } t>0, \\ (2)\quad (\nabla u\cdot\vec{n} =)\ \dfrac{\partial u}{\partial n} \overset{(5)}{=} 0 & \text{if } (x,y,z) \text{ belongs to the boundary of } C \text{ and } t\ge 0, \\ (3)\quad u(x,y,z,0) \overset{(5)}{=} \dfrac{50}{\sqrt{x^2+y^2}} & \text{if } (x,y,z) \text{ is in } C. \end{cases}$$

(The first equation says u obeys the heat equation, the second expresses the fact that the body is completely insulated, and the third gives the initial temperature distribution in the body.)

9 pts. (b)

$$\frac{dH}{dt} = \frac{d}{dt}\iiint_C c\rho u(x,y,z,t)\,dV = \iiint_C c\rho \frac{\partial u}{\partial t}(x,y,z,t)\,dV \overset{(1)}{=} \iiint_C c\rho k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)dV$$

$$= c\rho k\iiint_C \nabla\cdot(\nabla u)\,dV \overset{\text{Divergence Theorem}}{=} c\rho k\iint_{\partial C}\nabla u\cdot\vec{n}\,dS \overset{(2)}{=} c\rho k\iint_{\partial C} 0\,dS = 0.$$

Therefore the heat energy of the material in C is a constant function of t.

9 pts. (c) Let $U=\lim\limits_{t\to\infty} u(x,y,z,t)$ be the steady-state temperature of the material in C. Since $H(t)$ is constant,

$$H(0)=\lim_{t\to\infty} H(t) = \iiint_C c\rho \lim_{t\to\infty} u(x,y,z,t)\,dV = \iiint_C c\rho U\,dV = c\rho U\,\mathrm{vol}(C) = c\rho U 6\pi.$$

But $H(0)=\displaystyle\iiint_C c\rho u(x,y,z,0)\,dV = \int_0^2\int_0^{2\pi}\int_1^2 c\rho\,\frac{50}{r}\,r\,dr\,d\theta\,dz = c\rho 50\cdot 4\pi$.

Consequently, $U = \dfrac{c\rho 50\cdot 4\pi}{c\rho 6\pi} = \boxed{\dfrac{100}{3}}$.

#3. (a) $\mathcal{F}(g(\cdot - a))(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-a)e^{-i\xi x}\,dx$. Let $z = x-a$. Then $dz = dx$ so

4 pts.

(2) $$\mathcal{F}(g(\cdot - a))(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(z)e^{-i\xi(z+a)}\,dz = \frac{e^{-i\xi a}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(z)e^{-i\xi z}\,dz = e^{-i\xi a}\hat{g}(\xi).$$

(This formula holds for all absolutely integrable functions g on $(-\infty,\infty)$.)

Let $f(x) = \int_{-\infty}^{x} g(s)\,ds$ and $f'(x) = g(x)$ be absolutely integrable on $(-\infty,\infty)$.

Since f and f' are absolutely integrable on $(-\infty,\infty)$, an identity proved in lecture shows that $\mathcal{F}(f')(\xi) = i\xi\hat{f}(\xi)$ for all real ξ. That is,

(2) $$\frac{\hat{g}(\xi)}{i\xi} = \frac{\mathcal{F}(f')(\xi)}{i\xi} = \hat{f}(\xi) = \mathcal{F}\left(\int_{-\infty}^{\cdot} g(s)\,ds\right)(\xi) \quad \text{for all real } \xi \neq 0.$$

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29 pts (b) $u_{tt} - c^2u_{xx} = f(x,t)$, $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$ $(-\infty < x < \infty)$.

$$\mathcal{F}(u_{tt} - c^2u_{xx})(\xi) = \mathcal{F}(f(x,t))(\xi)$$

(3) $$\frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi) + c^2\xi^2\mathcal{F}(u)(\xi) = \hat{f}(\xi,t)$$ $\longleftarrow$ 2^{nd}-order ODE in t with parameter ξ.

$$\left(W = \begin{vmatrix} \cos(c\xi t) & \sin(c\xi t) \\ -c\xi\sin(c\xi t) & c\xi\cos(c\xi t)\end{vmatrix} = c\xi\right)$$

(3) $$\mathcal{F}(u)(\xi) = c_1(\xi)\cos(c\xi t) + c_2(\xi)\sin(c\xi t) + u_1(t)\cos(c\xi t) + u_2(t)\sin(c\xi t)$$

(3) where $u_1(t) = \int_0^t -\frac{\hat{f}(\xi,\tau)\sin(c\xi\tau)}{c\xi}\,d\tau$ and $u_2(t) = \int_0^t \frac{\hat{f}(\xi,\tau)\cos(c\xi\tau)}{c\xi}\,d\tau$

(3) $$\mathcal{F}(\varphi)(\xi) = \mathcal{F}(u)(\xi)\Big|_{t=0} = c_1(\xi) \quad \text{and} \quad \mathcal{F}(\psi)(\xi) = \mathcal{F}(u_t)(\xi)\Big|_{t=0} = c\xi c_2(\xi).$$

(3) $$\therefore \mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)\cos(c\xi t) + \frac{1}{c\xi}\mathcal{F}(\psi)(\xi)\sin(c\xi t) + \int_0^t \frac{\hat{f}(\xi,\tau)}{c\xi}\sin(c\xi(t-\tau))\,d\tau$$

Using the identities $\cos(c\xi t) = \frac{1}{2}e^{ic\xi t} + \frac{1}{2}e^{-ic\xi t}$ and $\sin(c\xi t) = \frac{1}{2i}e^{ic\xi t} - \frac{1}{2i}e^{-ic\xi t}$

15 pts. to here. and the transform facts $\mathcal{F}(g(x-a))(\xi) = \hat{g}(\xi)e^{-i\xi a}$ and $\mathcal{F}\left(\int_{-\infty}^{x} g(s)\,ds\right) = \frac{\hat{g}(\xi)}{i\xi}$ yields

15 pts. to here.

(3)
$$\mathcal{F}(u)(\xi) = \tfrac{1}{2}\mathcal{F}(\varphi)(\xi)e^{ic\xi t} + \tfrac{1}{2}\mathcal{F}(\varphi)(\xi)e^{-ic\xi t} + \tfrac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x}\psi(s)ds\right)(\xi)e^{ic\xi t} - \tfrac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x}\psi(s)ds\right)(\xi)e^{-ic\xi t}$$
$$+ \int_0^t \left[\tfrac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x} f(s,\tau)ds\right)(\xi)e^{ic\xi(t-\tau)} - \tfrac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x} f(s,\tau)ds\right)(\xi)e^{-ic\xi(t-\tau)}\right]d\tau$$

(3)
$$= \tfrac{1}{2}\mathcal{F}(\varphi(x+ct))(\xi) + \tfrac{1}{2}\mathcal{F}(\varphi(x-ct))(\xi) + \tfrac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x+ct}\psi(s)ds\right)(\xi) - \tfrac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x-ct}\psi(s)ds\right)(\xi)$$
$$+ \int_0^t \left[\tfrac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x+c(t-\tau)} f(s,\tau)ds\right)(\xi) - \tfrac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{x-c(t-\tau)} f(s,\tau)ds\right)(\xi)\right]d\tau$$

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(2)
$$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\tfrac{1}{2}[\varphi(x+ct) + \varphi(x-ct)]\right)(\xi) + \mathcal{F}\left(\tfrac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds\right)(\xi)$$
$$+ \int_0^t \mathcal{F}\left(\tfrac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)ds\right)(\xi)d\tau$$

Interchanging the order of integration in the last term and using linearity of the Fourier transform gives

(3)
$$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\tfrac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \tfrac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds + \tfrac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)ds\,d\tau\right)(\xi)$$

Applying the inversion formula leads to

(3)
$$\boxed{u(x,t) = \tfrac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \tfrac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds + \tfrac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)ds\,d\tau}.$$

29 pts. to here

4.(33 pts.) Solve the beam equation

$$u_{tt} + u_{xxxx} \overset{①}{=} 0 \text{ if } 0<x<1,\ 0<t<\infty,$$

subject to the boundary conditions

②-③-④-⑤ $$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0 \text{ if } t \ge 0,$$

and the initial conditions

$$u(x,0) \overset{⑥}{=} 2\sin(\pi x) - 3\sin(5\pi x) \text{ and } u_t(x,0) \overset{⑦}{=} 0 \text{ if } 0 \le x \le 1.$$

You may find useful the fact that the general solution to $y^{(4)} - \beta^4 y = 0$ is

$$y(x) = c_1 \cosh(\beta x) + c_2 \sinh(\beta x) + c_3 \cos(\beta x) + c_4 \sin(\beta x).$$

We use the method of separation of variables. We seek nontrivial solutions to the homogeneous portion of the problem, ①-②-③-④-⑤-⑦, of the form $u(x,t) = X(x)T(t)$. Substituting in ① gives

$$X(x)T''(t) + X^{(4)}(x)T(t) = 0 \quad \text{or} \quad \frac{-T''(t)}{T(t)} = \frac{X^{(4)}(x)}{X(x)} = \text{constant} = \lambda.$$ ②

Substituting in ② gives $X(0)T(t) = 0$ if $t \ge 0$ and since $u(x,t) = X(x)T(t)$ is not identically zero on $0 \le x \le 1$, $0 \le t < \infty$, it follows that $X(0) = 0$. Similar arguments using ③-④-⑤-⑦ leads to a coupled system of ODEs and BCs:

$$\begin{cases} X^{(4)}(x) - \lambda X(x) \overset{⑧}{=} 0, & X(0) \overset{⑨}{=} 0 \overset{⑩}{=} X(1),\ X''(0) \overset{⑪}{=} 0 \overset{⑫}{=} X''(1), \\ T''(t) + \lambda T(t) \overset{⑬}{=} 0, & T'(0) \overset{⑭}{=} 0. \end{cases}$$ ⑥ ④

We will assume that the eigenvalues λ of the eigenvalue problem ⑧-⑨-⑩-⑪-⑫ are real. (See Chapter 5 for the justification of this assumption.) We first claim that the eigenvalues λ are nonnegative. To see this, suppose that λ is an eigenvalue with corresponding eigenfunction $X = X(x)$. Then integrating by parts twice shows that

$$\lambda \int_0^1 X^2(x)\,dx = \int_0^1 X(x)(\lambda X(x))\,dx = \int_0^1 X(x) X^{(4)}(x)\,dx = \left(X(x)X^{(3)}(x) - X'(x)X''(x)\right)\Big|_0^1 + \int_0^1 (X''(x))^2\,dx$$

Now ⑨-⑩-⑪-⑫ imply that $\left(X(x)X^{(3)}(x) - X'(x)X''(x)\right)\Big|_0^1 = 0$ so

$$\lambda \int_0^1 X^2(x)\,dx = \int_0^1 (X''(x))^2\,dx \ge 0.$$

But $\int_0^1 X^2(x)\,dx > 0$ since $X = X(x)$ is not identically zero, and hence $\lambda \ge 0$ as claimed.

③

15 pts. to here

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Case $\lambda > 0$, say $\lambda = \beta^4$ where $\beta > 0$.

Then ⑧ becomes $X^{(4)}(x) - \beta^4 X(x) = 0$ and this has general solution $X(x) = c_1\cosh(\beta x) + c_2\sinh(\beta x) + c_3\cos(\beta x) + c_4\sin(\beta x)$ with $X''(x) = \beta^2 c_1\cosh(\beta x) + \beta^2 c_2\sinh(\beta x) - \beta^2 c_3\cos(\beta x) - \beta^2 c_4\sin(\beta x)$. ⑨ and ⑪ yield $0 = X(0) = c_1 + c_3$ and $0 = X''(0) = \beta^2 c_1 - \beta^2 c_3$ from which $c_1 = 0 = c_3$ follows. Applying ⑩ and ⑫ leads to $0 = X(1) = c_2\sinh(\beta) + c_4\sin(\beta)$ and $0 = X''(1) = \beta^2 c_2\sinh(\beta) - \beta^2 c_4\sin(\beta)$. Summing the equations gives $2c_2\sinh(\beta) = 0$ and hence $c_2 = 0$ since $\sinh(\beta) > 0$ for $\beta > 0$. Substituting then gives $c_4\sin(\beta) = 0$. The eigenvalue condition – i.e. the condition for the existence of a nontrivial solution – is $\sin(\beta) = 0$. Consequently the positive eigenvalues and corresponding eigenfunctions are

(4) ⑮ $\lambda_n = \beta_n^4 = (n\pi)^4$ and $X_n(x) = \sin(\beta_n x) = \sin(n\pi x)$ $\quad (n = 1, 2, 3, \ldots)$.

Case $\lambda = 0$. Then ⑧ becomes $X^{(4)}(x) = 0$ with general solution $X(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ and $X''(x) = 2c_2 + 6c_3 x$. Applying ⑨ and ⑪ yield $0 = X(0) = c_0$ and $0 = X''(0) = 2c_2$. Applying ⑫ and ⑩ then give $0 = X''(1) = 2c_2 + 6c_3$ (so $c_3 = 0$) and $0 = X(1) = c_1 + c_1 + c_2 + c_3$ (so $c_1 = 0$). Hence there are no nontrivial solutions in this case; i.e.

(2) zero is not an eigenvalue.

(2) Substituting from ⑮ into ⑬-⑭ and solving easily gives $T_n(t) = \cos(n^2\pi^2 t)$, up to a constant factor. The superposition principle yields that

(4) $$u(x,t) = \sum_{n=1}^{N} c_n X_n(x) T_n(t) = \sum_{n=1}^{N} c_n \sin(n\pi x)\cos(n^2\pi^2 t)$$

satisfies ①-②-③-④-⑤-⑦ for any integer $N \geq 1$ and any constants $c_1, c_2, \ldots, c_N$. We need to choose N and the c_ns so that ⑥ is satisfied:

(2) $$2\sin(\pi x) - 3\sin(5\pi x) = u(x,0) = \sum_{n=1}^{N} c_n \sin(n\pi x) \quad \text{for all } 0 \leq x \leq 1.$$

(2) By inspection, we may take $N = 5$, $c_1 = 2$, $c_5 = -3$, and all other $c_n = 0$. That is,

(2) $$\boxed{u(x,t) = 2\sin(\pi x)\cos(\pi^2 t) - 3\sin(5\pi x)\cos(25\pi^2 t)}$$

is a solution of ①-②-③-④-⑤-⑥-⑦. (Note: Energy techniques can be used to show that this is the unique solution of the problem.)

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#5. (a) We use separation of variables. We seek nontrivial solutions to the homogeneous portion of the problem

$$u_t - u_{xx} \overset{(1)}{=} 0 \quad \text{on} \quad 0 < x < 1,\ 0 < t < \infty,$$

$$u_x(0,t) \overset{(2)}{=} 0 \overset{(3)}{=} u(1,t) \quad \text{if} \quad t \geq 0,$$

$$u(x,0) \overset{(4)}{=} 1 - x^2 \quad \text{if} \quad 0 \leq x \leq 1,$$

1 of the form $u(x,t) = X(x)T(t)$. Substituting in (1) gives $X(x)T'(t) - X''(x)T(t) = 0$ so $\dfrac{-X''(x)}{X(x)} = \dfrac{-T'(t)}{T(t)} = \text{constant} = \lambda$. Substituting in (2) yields $X'(0)T(t) = 0$ if $t \geq 0$, and nontriviality of $u(x,t) = X(x)T(t)$ implies $X'(0) = 0$. A similar argument with (3) leads to $X(1) = 0$. Thus we have the coupled system of O.D.E.s and B.C.s:

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$$\begin{cases} X''(x) + \lambda X(x) \overset{(5)}{=} 0, \quad X'(0) \overset{(6)}{=} 0 \overset{(7)}{=} X(1), \\ T'(t) + \lambda T(t) \overset{(8)}{=} 0. \end{cases}$$

Note that (5)-(6)-(7) is the eigenvalue problem for the symmetric operator $T = \dfrac{-d^2}{dx^2}$ on $V = \{\varphi \in C^2[0,1] : \varphi'(0) = 0 = \varphi(1)\}$. It follows from problem 1 on Exam III that

1 the eigenvalues are real and nonnegative.

<u>Case $\lambda = 0$</u>: The general solution of (5) in this case, $X''(x) = 0$, is $X(x) = c_1 x + c_2$. Then $0 = X'(0) = c_1$ and $0 = X(1) = c_1 + c_2$ so $c_1 = 0 = c_2$. Thus $\lambda = 0$ is not

1 an eigenvalue.

<u>Case $\lambda > 0$</u>, say $\lambda = \alpha^2$ where $\alpha > 0$. The general solution of (5) in this case, $X''(x) + \alpha^2 X(x) = 0$, is $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. Then $X'(x) = -\alpha c_1 \sin(\alpha x) + \alpha c_2 \cos(\alpha x)$ so $0 = X'(0) = \alpha c_2$ implies $c_2 = 0$. Also $0 = X(1) = c_1 \cos(\alpha) + c_2 \sin(\alpha) = c_1 \cos(\alpha)$

1 so $\cos(\alpha) = 0$ is the condition for a nontrivial solution. Hence $\alpha = \alpha_n = (2n+1)\frac{\pi}{2}$ for $n = 0, 1, 2, \ldots$ Consequently the eigenvalues and eigenfunctions of (5)-(6)-(7) are

3 $$\lambda_n = \alpha_n^2 = \frac{(2n+1)^2\pi^2}{4} \quad \text{and} \quad X_n(x) = \cos(\alpha_n x) = \cos\left((2n+1)\frac{\pi x}{2}\right) \qquad (n = 0, 1, 2, \ldots)$$

Substituting $\lambda = \lambda_n = (2n+1)^2\pi^2/4$ in (8) gives $T_n'(t) + \dfrac{(2n+1)^2\pi^2}{4} T_n(t) = 0$

13 pts. to here.

13 pts. to here.

(1) which has general solution $T_n(t) = c_n e^{-\frac{(2n+1)^2\pi^2}{4}t}$. Then $u_n(x,t) = X_n(x)T_n(t)$

(1) $= c_n \cos\left(\frac{(2n+1)\pi x}{2}\right) e^{-\frac{(2n+1)^2\pi^2}{4}t}$ $(n=0,1,2,\dots)$ are solutions to ①-②-③. The superposition principle then implies that

(4)
$$u(x,t) \overset{⑨}{=} \sum_{n=0}^{\infty} c_n \cos\left(\frac{(2n+1)\pi x}{2}\right) e^{-\frac{(2n+1)^2\pi^2}{4}t}$$

is a formal solution of ①-②-③ for any constants $c_0, c_1, c_2, \dots$ To solve ④ as well, we need

(1)
$$1 - x^2 = u(x,0) \overset{⑩}{=} \sum_{n=0}^{\infty} c_n \cos\left(\frac{(2n+1)\pi x}{2}\right) \qquad \text{if } 0 \le x \le 1.$$

Thus the constants c_n $(n=0,1,2,\dots)$ should be the Fourier coefficients of $f(x) = 1-x^2$ with respect to the orthogonal set $\Phi = \left\{ \cos\left(\frac{(2n+1)\pi x}{2}\right) : n=0,1,\dots \right\}$ on $[0,1]$. We compute

(3)
$$c_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} \overset{⑪}{=} \frac{\int_0^1 f(x)\cos((2n+1)\pi x/2)\,dx}{\int_0^1 \cos^2((2n+1)\pi x/2)\,dx} = \frac{32(-1)^n}{(2n+1)^3\pi^3} \qquad (n=0,1,2,\dots)$$

(2) and the Fourier series of f with respect to Φ converges uniformly to f on $[0,1]$. (See problem 2 of Exam III for details.) That is, ⑩ holds when c_n is given by ⑪ for $n=0,1,2,\dots$. Hence, substituting in ⑨ yields the following solution to ①-②-③-④:

(1)
$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \frac{32(-1)^n \cos((2n+1)\pi x/2)\, e^{-\frac{(2n+1)^2\pi^2}{4}t}}{(2n+1)^3\pi^3}}.$$

25 pts. to here.

(1) (b) The solution to part (a) is unique. To see this we use energy considerations. Let $v = v(x,t)$ be another solution to the problem in part (a) and consider

(1) $W(x,t) = u(x,t) - v(x,t)$ where $u = u(x,t)$ is the solution above. Then W

27 pts. to here

27 pts. to here.

solves

(1) $$\begin{cases} w_t - w_{xx} \overset{(12)}{=} 0 & \text{on } 0<x<1,\ 0<t<\infty, \\ w_x(0,t) \overset{(13)}{=} 0 \overset{(14)}{=} w(1,t) & \text{if } t \geq 0 \\ w(x,0) \overset{(15)}{=} 0 & \text{if } 0 \leq x \leq 1. \end{cases}$$

The energy function of w is given by

(1) $$E(t) = \int_0^1 w^2(x,t)\,dx$$

for $t \geq 0$. Then

(1) $$\frac{dE}{dt} = \frac{d}{dt}\int_0^1 w^2(x,t)\,dx = \int_0^1 \frac{\partial}{\partial t}\left(w^2(x,t)\right)dx = 2\int_0^1 w(x,t)w_t(x,t)\,dx.$$

Applying (12) we find, after an integration by parts, that

(1) $$\frac{dE}{dt} = 2\int_0^1 \overbrace{w(x,t)}^{v}\overbrace{w_{xx}(x,t)\,dx}^{dV} = 2w(x,t)w_x(x,t)\Big|_0^1 - 2\int_0^1 w_x^2(x,t)\,dx.$$

Since $w(1,t) \overset{(13)}{=} 0 \overset{(14)}{=} w_x(0,t)$ for all $t \geq 0$, the boundary terms vanish, giving

(1) $$\frac{dE}{dt} = -2\int_0^1 w_x^2(x,t)\,dx \leq 0 \quad \text{if } t \geq 0.$$

That is, $0 \leq E(t) \leq E(0) = \int_0^1 w^2(x,0)\,dx \overset{\text{by (15)}}{=} \int_0^1 0\,dx = 0$ for all $t \geq 0$.

(1) Thus, $E(t) = 0$ for all $t \geq 0$. The vanishing theorem then implies $w^2(x,t) = 0$ for all $0 \leq x \leq 1$ and each $t \geq 0$. Therefore

(1) $$0 = w(x,t) = u(x,t) - v(x,t)$$

for all $0 \leq x \leq 1$ and all $t \geq 0$; i.e. the solution $u = u(x,t)$ to (1)-(2)-(3)-(4) is unique.

34 pts. to here.

6. (34 pts.) (a) Find a solution to $\nabla^2 u = 0$ in the cube C: $0<x<1$, $0<y<1$, $0<z<1$, subject to the boundary conditions $u(x,y,1)=\sin(\pi x)\sin^3(\pi y)$ for $0\le x\le 1$, $0\le y\le 1$ and $u=0$ on the other five faces of C.
(b) State the maximum/minimum principle for harmonic functions and use it to show that the problem in part (a) has only one solution.

We must solve $u_{xx}+u_{yy}+u_{zz} \overset{(1)}{=} 0$ in C subject to $u(0,y,z) \overset{(2)}{=} 0 \overset{(3)}{=} u(1,y,z)$ for $0\le y\le 1$, $0\le z\le 1$, $u(x,0,z) \overset{(4)}{=} 0 \overset{(5)}{=} u(x,1,z)$ for $0\le x\le 1$, $0\le z\le 1$, $u(x,y,0) \overset{(6)}{=} 0$ and $u(x,y,1) \overset{(7)}{=} \sin(\pi x)\sin^3(\pi y)$ for $0\le x\le 1$, $0\le y\le 1$. We seek nontrivial solutions of the
1 homogeneous portion of this problem, (1)–(6), of the form $u(x,y,z)=X(x)Y(y)Z(z)$. Substituting in (1) yields $X''(x)Y(y)Z(z)+X(x)Y''(y)Z(z)+X(x)Y(y)Z''(z)=0$ in C. Dividing through by $X(x)Y(y)Z(z)$ and rearranging gives

$$\frac{Y''(y)}{Y(y)}+\frac{Z''(z)}{Z(z)} = -\frac{X''(x)}{X(x)} = \text{constant} \overset{(8)}{=} \lambda$$

and then

$$\frac{Z''(z)}{Z(z)}-\lambda = -\frac{Y''(y)}{Y(y)} = \text{constant} \overset{(9)}{=} \mu .$$

Substituting $u=XYZ$ in (2) and (3) give $X(0)Y(y)Z(z)=0=X(1)Y(y)Z(z)$ for all $0\le y\le 1$, $0\le z$. In order that the solution u be nontrivial we must have $X(0) \overset{(10)}{=} 0 \overset{(11)}{=} X(1)$. Similar arguments lead to $Y(0) \overset{(12)}{=} 0 \overset{(13)}{=} Y(1)$ and $Z(0) \overset{(14)}{=} 0$ (using (4), (5), and (6), respectively). Thus we are led to the coupled system of ODEs and BCs using (8), (9), (10), (11), (12):

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$$(*)\quad \begin{cases} X''(x)+\lambda X(x)=0, & X(0)=0=X(1), \\ Y''(y)+\mu Y(y)=0, & Y(0)=0=Y(1), \\ Z''(z)-(\lambda+\mu)Z(z)=0, & Z(0)=0 . \end{cases}$$

The first two lines of $(*)$ are eigenvalue problems for the operators $-\frac{d^2}{dx^2}$ and $-\frac{d^2}{dy^2}$, respectively, with Dirichlet boundary conditions. The eigenvalues and eigenfunctions are then well-known:

3 $\lambda_\ell = (\ell\pi)^2$, $X_\ell(x)=\sin(\ell\pi x)$ $(\ell=1,2,3,\dots)$,

3 $\mu_m = (m\pi)^2$, $Y_m(y)=\sin(m\pi y)$ $(m=1,2,3,\dots)$.

Substituting $\lambda=\lambda_\ell$ and $\mu=\mu_m$ in the last line of $(*)$ we find the general solution
1 of the ODE is $Z_{\ell,m}(z)=c_1\cosh(\pi\sqrt{\ell^2+m^2}\,z)+c_2\sinh(\pi\sqrt{\ell^2+m^2}\,z)$. Applying the BC $Z_{\ell,m}(0)=0$
1 means that $c_1=0$ so $Z_{\ell m}(z)=\sinh(\pi z\sqrt{\ell^2+m^2})$, up to a constant multiple.

20 pts to here

20 pts. to here.

(1) Therefore $u_{\ell,m}(x,y,z) = X_\ell(x) Y_m(y) Z_{\ell,m}(z) = \sin(\ell\pi x)\sin(m\pi y)\sinh\left(\pi z\sqrt{\ell^2+m^2}\right)$

($\ell = 1,2,3,\ldots$ and $m=1,2,3,\ldots$) solves (1)–(6). By the superposition principle, a formal solution of (1)–(6) is

(4)
$$u(x,y,z) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} a_{\ell,m}\sin(\ell\pi x)\sin(m\pi y)\sinh\left(\pi z\sqrt{\ell^2+m^2}\right)$$

for arbitrary constants $a_{\ell,m}$ ($\ell=1,2,3,\ldots$ and $m=1,2,3,\ldots$). We need to choose the constants so that (7) is satisfied:

$$\sin(\pi x)\sin^3(\pi y) = u(x,y,1) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} a_{\ell,m}\sin(\ell\pi x)\sin(m\pi y)\sinh\left(\pi\sqrt{\ell^2+m^2}\right)$$

for all $0\le x\le 1$, $0\le y\le 1$. Using the identity $\sin^3(A) = \frac{3}{4}\sin(A) - \frac{1}{4}\sin(3A)$ this can be rewritten as

(1)
$$\frac{3}{4}\sin(\pi x)\sin(\pi y) - \frac{1}{4}\sin(\pi x)\sin(3\pi y) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} a_{\ell,m}\sin(\ell\pi x)\sin(m\pi y)\sinh\left(\pi\sqrt{\ell^2+m^2}\right).$$

(1) By inspection, $\frac{3}{4} = a_{1,1}\sinh(\pi\sqrt{2})$, $-\frac{1}{4} = a_{1,3}\sinh(\pi\sqrt{10})$, and all other $a_{\ell,m}=0$.

Therefore

(2)
$$\boxed{u(x,y,z) = \frac{3\sin(\pi x)\sin(\pi y)\sinh(\pi z\sqrt{2})}{4\sinh(\pi\sqrt{2})} - \frac{\sin(\pi x)\sin(3\pi y)\sinh(\pi z\sqrt{10})}{4\sinh(\pi\sqrt{10})}}$$

is a continuous solution of the problem (1)–(7).

29 pts. to here.

(b) Maximum/Minimum Principle: Let $u = u(x,y,z)$ be a solution to

(2) Laplace's equation $\nabla^2 u = 0$ in a bounded, open set R of $\mathbb{R}^3$ and let u be continuous on the closure $\overline{R} = R \cup \partial R$ of R. Then

$$\max\{u(x,y,z) : (x,y,z)\in\overline{R}\} = \max\{u(x,y,z) : (x,y,z)\in\partial R\}$$ and

$$\min\{u(x,y,z) : (x,y,z)\in\overline{R}\} = \min\{u(x,y,z) : (x,y,z)\in\partial R\}.$$

31 pts. to here

Let $u = u(x,y,z)$ denote the (continuous) solution to the problem ①-⑦ obtained in part (a), and let $u = v(x,y,z)$ be another solution to ①-⑦ that is continuous on $\overline{C} = C \cup \partial C$: $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

(1) Define w on $\overline{C}$ by setting $w(x,y,z) = u(x,y,z) - v(x,y,z)$. Then w is a

(1) continuous function on $\overline{C}$ such that $\nabla^2 w = 0$ in C and $w = 0$ on ∂C.

(1) By the maximum/minimum principle for harmonic functions, $w = 0$ on $\overline{C}$. That is, $u(x,y,z) = v(x,y,z)$ for all (x,y,z) in $\overline{C}$. Thus the solution to the problem in part (a) is unique.

34 pts. to here.

Math 325

Final Exam

Fall 2012

$n = 32$

mean = 122.1

median = 119

standard deviation = 43.9

Distribution of Scores

Range	Graduate Grade	Undergraduate Grade	Frequency
174 – 200	A	A	4
146 – 173	B	B	6
120 – 145	C	B	6
100 – 119	C	C	7
0 – 99	F	D	9