Mathematics 5325 | Final Exam Summer 2015 | Name: Dr. Grow

1.(28 pts.) (a) Show that $u(x,y)=e^{y-x}$ is a particular solution to the nonhomogeneous first order partial differential equation $e^x u_x + xyu_y = (xye^{-x}-1)e^y$.

(b) Find the general solution of $e^x u_x + xyu_y = (xye^{-x}-1)e^y$.

2.(28 pts.) Classify the type (parabolic, hyperbolic, or elliptic) of the second order partial differential equation $u_{xx}-4u_{xy}+3u_{yy}=\cos(x+y)$ and, if possible, find the general solution in the xy-plane.

3.(28 pts.) Solve $u_t - u_{xx}=0$ in the upper half-plane $-\infty<x<\infty,\ 0<t<\infty$, subject to the initial condition $u(x,0)=x^3$ if $-\infty<x<\infty$. You may find the following identities useful:

$$2\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}\,,\ \int_{-\infty}^{\infty} p^3 e^{-p^2}\,dp = \int_{-\infty}^{\infty} pe^{-p^2}\,dp = 0.$$

4.(29 pts.) In this problem, you may find useful the fact that the Laplacian of u in spherical coordinates is given by

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right)+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2}.$$

A spherical wave is a solution of the three-dimensional wave equation of the form $u=u(r,t)$, independent of the spherical coordinates φ and θ; here r denotes the distance from the origin.

(a) Show that a spherical wave satisfies the partial differential equation $u_{tt}=c^2\left(u_{rr}+\frac{2}{r}u_r\right)$.

(b) Make the change of variables $v(r,t)=ru(r,t)$ in the partial differential equation in (a) and show that v satisfies the partial differential equation $v_{tt}-c^2v_{rr}=0$.

(c) Find the spherical wave $u=u(r,t)$ satisfying the initial conditions $u(r,0)=r^3$ and $u_t(r,0)=4cr^2$ for all $r>0$. (Hint: Use part (b).)

5.(29 pts.) Use Fourier transform techniques to derive a formula for the solution to the initial value problem:

$$u_{tt}-u_{xx}=f(x,t) \text{ if } -\infty<x<\infty,\ 0<t<\infty,$$
$$u(x,0)=0 \text{ and } u_t(x,0)=0 \text{ if } -\infty<x<\infty.$$

6.(29 pts.) (a) Find a solution to the damped wave equation

(1) $$u_{tt}-u_{xx}+2u_t=0 \text{ in } 0<x<\pi,\ 0<t<\infty,$$

satisfying

(2)-(3) $$u_x(0,t)=0=u_x(\pi,t) \text{ for } t\ge 0,$$

(4)-(5) $$u(x,0)=0 \text{ and } u_t(x,0)=x^2 \text{ for } 0\le x\le\pi.$$

(b) Show that the energy $E(t)=\int_0^{\pi}\left[\frac{1}{2}u_t^2(x,t)+\frac{1}{2}u_x^2(x,t)\right]dx$ of a solution to (1)-(2)-(3) is a decreasing function for $t\ge 0$.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$
A.	$\begin{cases}1 & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\dfrac{2}{\pi}}\dfrac{\sin(b\xi)}{\xi}$
B.	$\begin{cases}1 & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\dfrac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\dfrac{1}{x^2+a^2} \quad (a>0)$	$\sqrt{\dfrac{\pi}{2}}\dfrac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases}x & \text{if } 0<x\le b,\\ 2b-x & \text{if } b<x<2b,\\ 0 & \text{otherwise.}\end{cases}$	$\dfrac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases}e^{-ax} & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$	$\dfrac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases}e^{ax} & \text{if } b<x<c,\\ 0 & \text{otherwise.}\end{cases}$	$\dfrac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases}e^{iax} & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\dfrac{2}{\pi}}\dfrac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases}e^{iax} & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\dfrac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a>0)$	$\dfrac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\dfrac{\sin(ax)}{x} \quad (a>0)$	$\begin{cases}0 & \text{if } \lvert\xi\rvert\ge a,\\ \sqrt{\dfrac{\pi}{2}} & \text{if } \lvert\xi\rvert<a.\end{cases}$

#1. (a) $u_p(x,y) = e^{y-x}$ so $\frac{\partial u_p}{\partial x} = -e^{y-x}$ and $\frac{\partial u_p}{\partial y} = e^{y-x}$. Therefore

$$e^x \frac{\partial u_p}{\partial x} + xy\frac{\partial u_p}{\partial y} = e^x\left(-e^{y-x}\right) + xye^{y-x} = -e^y + xye^{y-x} \stackrel{\checkmark}{=} (xye^{-x}-1)e^y.$$

(b) The general solution of $e^x u_x + xy u_y \overset{①}{=} (xye^{-x}-1)e^y$ is $u(x,y) = u_h(x,y) + u_p(x,y)$ where u_h is the general solution of the associated homogeneous equation $e^x u_x + xy u_y \overset{②}{=} 0$ and u_p is the particular solution of the nonhomogeneous equation from part (a). The characteristic curves of $a(x,y)u_x + b(x,y)u_y = 0$ are given by $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$. Thus $\frac{dy}{dx} = \frac{xy}{e^x}$ are the characteristic curves of ②. Separating variables gives

$$\frac{dy}{y} = xe^{-x}dx \text{ so integrating yields } \ln|y| = \int \frac{dy}{y} = \int \underbrace{x}_{v}\,\underbrace{e^{-x}dx}_{dv} = -xe^{-x} - \int -e^{-x}dx$$

$$= -xe^{-x} - e^{-x} + c = c - (x+1)e^{-x}.$$ Consequently $y(x) \overset{③}{=} A e^{-(x+1)e^{-x}}$ where $A = \pm e^c$ is an arbitrary constant. Solutions to ② are constant along characteristic curves so along such a curve a solution $u = u_h(x,y)$ satisfies

$$u_h(x, y(x)) = u_h\left(x, Ae^{-(x+1)e^{-x}}\right) = u_h(-1, Ae^0) = f(A).$$

But curves of the form ③ "pave" the xy-plane, so given an arbitrary point (x,y) in the plane,

$$u_h(x,y) = f(A) = f\left(y e^{(x+1)e^{-x}}\right)$$

where f is an arbitrary C^1-function of a single real variable. Thus, the general solution of ① is

$$u(x,y) = u_h(x,y) + u_p(x,y)$$

$$\boxed{u(x,y) = f\left(ye^{(x+1)e^{-x}}\right) + e^{y-x}}.$$

#2. $u_{xx} - 4u_{xy} + 3u_{yy} \overset{①}{=} \cos(x+y)$. $\quad B^2 - 4AC = (-4)^2 - 4(1)(3) = 4 > 0$

so ① is a $\boxed{\text{hyperbolic}}$ PDE. We rewrite ① in an equivalent form with the operator in factored form:

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)u = \left(\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x \partial y} + 3\frac{\partial^2}{\partial y^2}\right)u \overset{①'}{=} \cos(x+y)$$

This suggests the change-of-variables

$$\xi = -(\beta x - \alpha y) = -(-x-y) = x+y,$$
$$\eta = -(\delta x - \gamma y) = -(-3x-y) = 3x+y.$$

If v is a C^1-function of two independent variables x and y then:

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\overset{1}{\frac{\partial \xi}{\partial x}} + \frac{\partial v}{\partial \eta}\overset{3}{\frac{\partial \eta}{\partial x}} = \frac{\partial v}{\partial \xi} + 3\frac{\partial v}{\partial \eta} \quad \text{so} \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + 3\frac{\partial}{\partial \eta},$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\overset{1}{\frac{\partial \xi}{\partial y}} + \frac{\partial v}{\partial \eta}\overset{1}{\frac{\partial \eta}{\partial y}} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \quad \text{so} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

Substituting in ①' the operator identities and the first equation in the change-of-variables gives

$$\left[\left(\cancel{\frac{\partial}{\partial \xi}} + 3\frac{\partial}{\partial \eta}\right) - \left(\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right)\right]\left[\left(\frac{\partial}{\partial \xi} + \cancel{3\frac{\partial}{\partial \eta}}\right) - 3\left(\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}}\right)\right]u = \cos(\xi)$$

$$2\frac{\partial}{\partial \eta}\left(-2\frac{\partial}{\partial \xi}\right)u = \cos(\xi)$$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = -\frac{1}{4}\cos(\xi).$$

Integrating with respect to η, holding ξ fixed, leads to

$$\frac{\partial u}{\partial \xi} = \int -\frac{1}{4}\cos(\xi)\,d\eta = -\frac{1}{4}\eta\cos(\xi) + c_1(\xi).$$

Integrating again with respect to ξ, holding η fixed, yields

$$u = \int\left[-\frac{1}{4}\eta\cos(\xi) + c_1(\xi)\right]d\xi = -\frac{1}{4}\eta\sin(\xi) + \int c_1(\xi)\,d\xi + c_2(\eta).$$

Therefore $\boxed{u(x,y) = -\frac{1}{4}(3x+y)\sin(x+y) + f(x+y) + g(3x+y)}$ where f and g are C^2-functions of a single real variable.

#3. $u_t - u_{xx} \overset{①}{=} 0$ if $-\infty < x < \infty,\ 0 < t < \infty$, subject to $u(x,0) \overset{②}{=} x^3$ if $-\infty < x < \infty$.

A solution to the diffusion equation in the upper half-plane satisfing $u(x,0) = \varphi(x)$ on the x-axis is given by

$$(*) \qquad u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy, \qquad (t>0).$$

In our case $k = 1$ and $\varphi(x) = x^3$, so a candidate for a solution to ①-② is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^3\,dy.$$

Let $p = \frac{y-x}{\sqrt{4t}}$. Then $dp = \frac{dy}{\sqrt{4t}}$, $y \to +\infty$ implies $p \to +\infty$ and $y \to -\infty$ implies $p \to -\infty$.

Therefore

$$u(x,t) = \frac{1}{\sqrt{\cancel{4}\pi \cancel{t}}} \int_{-\infty}^{\infty} e^{-p^2} (x + p\sqrt{4t})^3 \cancel{\sqrt{4t}}\,dp$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \left[x^3 + 3x^2 p\sqrt{4t} + 3x(p\sqrt{4t})^2 + (p\sqrt{4t})^3 \right] dp$$

$$= \overset{1}{\cancel{\frac{x^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp}} + \overset{0}{\cancel{\frac{3x^2\sqrt{4t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p\,dp}} + \overset{1/2}{\cancel{\frac{3x4t}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^2\,dp}} + \overset{0}{\cancel{\frac{(4t)^{3/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^3\,dp}}$$

$$u(x,t) = x^3 + 6xt$$

Since $(*)$ gives a solution when φ is continuous and bounded, and in our case $\varphi(x) = x^3$ is continuous but <u>not</u> bounded, we must "check" our candidate for a solution to ①-②.

①? $u_t - u_{xx} = \frac{\partial}{\partial t}(x^3 + 6xt) - \frac{\partial^2}{\partial x^2}(x^3 + 6xt) = 6x - 6x \overset{\checkmark}{=} 0$ if $-\infty < x < \infty,\ 0 < t < \infty$.

②? $u(x,0) = x^3 - 6x(0) \overset{\checkmark}{=} x^3$ if $-\infty < x < \infty$.

Therefore $\boxed{u(x,t) = x^3 + 6xt}$ solves ①-②.

(∘ = 3.5)

4. (a) If $u = u(r,t)$, independent of φ and θ, then $\frac{\partial u}{\partial \varphi} = 0 = \frac{\partial^2 u}{\partial \theta^2}$.

Therefore, since u satisfies the 3-D wave equation,

$$0 = u_{tt} - c^2 \nabla^2 u$$
$$= u_{tt} - c^2\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi \cancelto{0}{\frac{\partial u}{\partial \varphi}}\right) + \frac{1}{r^2 \sin^2\varphi}\cancelto{0}{\frac{\partial^2 u}{\partial \theta^2}}\right).$$

Consequently,

$$u_{tt} = c^2\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)\right) = c^2\left(\frac{1}{r^2}\left(r^2 \frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r}\right)\right)$$

(2) so $$\boxed{u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right).}$$

(b) Let u be a spherical wave and let $v(r,t) = r u(r,t)$. Then

$v_{tt} = r u_{tt}$, $v_r = r u_r + u$, and $v_{rr} = r u_{rr} + 2u_r$. Consequently

(3) $$v_{tt} - c^2 v_{rr} = r u_{tt} - c^2\left(r u_{rr} + 2u_r\right) = r\Big[\underbrace{u_{tt} - c^2\left(u_{rr} + \tfrac{2}{r}u_r\right)}_{0 \text{ by part (a)}}\Big] \overset{\checkmark}{=} 0,$$

at least when $r > 0$.

4 (c) Let $u = u(r,t)$ be a spherical wave satisfying $u(r,0) = r^3$ and $u_t(r,0) = 4cr^2$ for all $r > 0$. Let $v(r,t) = r\,u(r,t)$. Then v satisfies

$$v_{tt} - c^2 v_{rr} = 0 \quad \text{if } 0 < r < \infty,\ 0 < t < \infty,$$
$$v(r,0) = r^4 \text{ and } v_t(r,0) = 4cr^3 \quad \text{if } 0 < r < \infty.$$

The d'Alembert formula, $v(x,t) = \frac{1}{2}\left[\varphi(x-ct) + \varphi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$

gives a solution to $v_{tt} - c^2 v_{xx} = 0$ if $-\infty < x < \infty$, $0 < t < \infty$, subject to $v(x,0) = \varphi(x)$ and $v_t(x,0) = \psi(x)$ if $-\infty < x < \infty$. In our case,

$\varphi(x) = x^4$ and $\psi(x) = 4cx^3$ so

#4. (cont.)

$$v(r,t) = \frac{1}{2}\left[(r-ct)^4 + (r+ct)^4\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} 4cs^3\,ds$$

$$= \frac{1}{2}\left[(r-ct)^4 + (r+ct)^4\right] + \frac{1}{2}\left[s^4 \Big|_{s=r-ct}^{s=r+ct} \right]$$

$$= \frac{1}{2}\left[\cancel{(r-ct)^4} + (r+ct)^4\right] + \frac{1}{2}\left[(r+ct)^4 - \cancel{(r-ct)^4}\right]$$

$$= (r+ct)^4 \quad \left(= r^4 + 4r^3ct + 6r^2c^2t^2 + 4rc^3t^3 + c^4t^4 \right)$$

Therefore $u(r,t) = \frac{1}{r}v(r,t) = \frac{(r+ct)^4}{r}$ is a candidate for a solution to $u_{tt} \overset{①}{=} c^2\left(u_{rr} + \frac{2}{r}u_r\right)$ in $0<r<\infty$, $0<t<\infty$, subject to the initial conditions $u(r,0) \overset{②}{=} r^3$ and $u_t(r,0) \overset{③}{=} 4cr^2$ if $r>0$.

<u>Check</u>: $$u_{tt} - c^2\left(u_{rr} + \frac{2}{r}u_r\right) = \frac{12}{r}(r+ct)^2c^2 - c^2\left[6r + 8ct + \cancel{\frac{2c^4t^4}{r^3}} + \frac{2}{r}\left(3r^2 + 8rct + 6c^2t^2 - \cancel{\frac{c^4t^4}{r^2}}\right)\right]$$

$$= c^2\left[12r + 24ct + \frac{12c^2t^2}{r} - 6r - 8ct - 6r - 16ct - \frac{12c^2t^2}{r}\right]$$

$$\overset{\checkmark}{=} 0 \qquad (0<r<\infty,\ 0<t<\infty)$$

$$u(r,0) = \frac{1}{r}(r+0)^4 \overset{\checkmark}{=} r^3 \qquad (0<r<\infty)$$

$$u_t(r,t) = \frac{4}{r}(r+ct)^3c \qquad (0<r<\infty,\ 0<t<\infty)$$

$$u_t(r,0) = \frac{4}{r}(r+0)^3c \overset{\checkmark}{=} 4cr^2 \qquad (0<r<\infty)$$

Therefore $\boxed{u(r,t) = \frac{(r+ct)^4}{r}}$ solves ①-②-③ for $0<r<\infty$, $0<t<\infty$.

#5. Suppose $u = u(x,t)$ solves

$$u_{tt}(x,t) - u_{xx}(x,t) \overset{①}{=} f(x,t) \quad \text{if} \quad -\infty < x < \infty, \; 0 < t < \infty,$$

subject to

$$u(x,0) \overset{②}{=} 0 \overset{③}{=} u_t(x,0) \quad \text{if} \quad -\infty < x < \infty .$$

Take the Fourier transform of ① with respect to x holding t fixed:

$$\mathcal{F}\big(u_{tt}(\cdot,t) - u_{xx}(\cdot,t)\big)(\xi) = \mathcal{F}\big(f(\cdot,t)\big)(\xi).$$

Using properties of Fourier transforms and interchanging differentiation with respect to t (twice) and integration with respect to x over $(-\infty,\infty)$ (in the Fourier transform of the first term of the left member above) gives

$$\frac{\partial^2}{\partial t^2}\mathcal{F}\big(u(\cdot,t)\big)(\xi) + \xi^2 \mathcal{F}\big(u(\cdot,t)\big)(\xi) \overset{④}{=} \hat{f}(\xi,t).$$

The general solution of this ODE in the variable t (with parameter ξ) is

$$\mathcal{F}\big(u(\cdot,t)\big)(\xi) \overset{⑤}{=} U_h(\xi,t) + U_p(\xi,t)$$

where U_h is the general solution of the associated homogeneous ODE

$$\frac{\partial^2}{\partial t^2}U(\xi,t) + \xi^2 U(\xi,t) \overset{⑥}{=} 0$$

and U_p is a particular solution of ④. It is straightforward that

$$U_h(\xi,t) \overset{⑦}{=} c_1(\xi)e^{i\xi t} + c_2(\xi)e^{-i\xi t}$$

is the general solution of ⑥. Variation of parameters shows that

$$U_p(\xi,t) = e^{i\xi t}\int \frac{-\hat{f}(\xi,t)e^{-i\xi t}}{W(e^{i\xi t}, e^{-i\xi t})}\,dt + e^{-i\xi t}\int \frac{\hat{f}(\xi,t)e^{i\xi t}}{W(e^{i\xi t}, e^{-i\xi t})}\,dt$$

where the Wronskian of $e^{i\xi t}$ and $e^{-i\xi t}$ is

#5 (cont.)

$$W\left(e^{i\xi t}, e^{-i\xi t}\right) = \begin{vmatrix} e^{i\xi t} & e^{-i\xi t} \\ i\xi e^{i\xi t} & -i\xi e^{-i\xi t} \end{vmatrix} = -2i\xi. \quad \text{Thus}$$

$$U_p(\xi,t) = e^{i\xi t}\int \frac{\hat{f}(\xi,t)e^{-i\xi t}}{2i\xi}\,dt - e^{-i\xi t}\int \frac{\hat{f}(\xi,t)e^{i\xi t}}{2i\xi}\,dt$$

$$\overset{⑧}{=} e^{i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{-i\xi\tau}}{2i\xi}\,d\tau - e^{-i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{i\xi\tau}}{2i\xi}\,d\tau$$

$$= \int_0^t \frac{\hat{f}(\xi,\tau)}{\xi}\left[\frac{e^{i\xi(t-\tau)} - e^{-i\xi(t-\tau)}}{2i}\right]d\tau$$

$$\overset{⑨}{=} \int_0^t \hat{f}(\xi,\tau)\,\frac{\sin(\xi(t-\tau))}{\xi}\,d\tau .$$

Substituting from ⑦ and ⑧ into ⑤ yields

$$\mathcal{F}\big(u(\cdot,t)\big)(\xi) \overset{⑩}{=} c_1(\xi)e^{i\xi t} + c_2(\xi)e^{-i\xi t} + e^{i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{-i\xi\tau}}{2i\xi}\,d\tau - e^{-i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{i\xi\tau}}{2i\xi}\,d\tau$$

Applying the initial condition ② gives

$$0 \overset{⑪}{=} \mathcal{F}\big(u(\cdot,0)\big)(\xi) = c_1(\xi) + c_2(\xi).$$

Differentiating ⑩ yields

$$\mathcal{F}\big(u_t(\cdot,t)\big)(\xi) = \frac{\partial}{\partial t}\mathcal{F}\big(u(\cdot,t)\big)(\xi)$$

$$\overset{⑫}{=} i\xi c_1(\xi)e^{i\xi t} - i\xi c_2(\xi)e^{-i\xi t} + i\xi e^{i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{-i\xi\tau}}{2i\xi}\,d\tau + \cancel{e^{i\xi t}}\,\frac{\hat{f}(\xi,t)\cancel{e^{-i\xi t}}}{2i\xi}$$

$$+ i\xi e^{-i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{i\xi\tau}}{2i\xi}\,d\tau - \cancel{e^{-i\xi t}}\,\frac{\hat{f}(\xi,t)\cancel{e^{i\xi t}}}{2i\xi}.$$

Evaluating ⑫ at $t=0$ and using the initial condition ③ leads to

#5 (cont.)

$$0 \overset{(13)}{=} \mathcal{F}(u_t(\cdot,0))(\xi) = i\xi c_1(\xi) - i\xi c_2(\xi)$$

It is easy to see that the solution to the system (11)-(13) is $c_1(\xi) = 0 = c_2(\xi)$. Substituting these results into (10) and using (8) and (9) produces

$$\mathcal{F}(u(\cdot,t))(\xi) \overset{(14)}{=} \int_0^t \frac{\hat{f}(\xi,\tau)\sin(\xi(t-\tau))}{\xi}\,d\tau .$$

Taking $b = t-\tau$ in formula A in the Fourier transform table,

$$\mathcal{F}\left(\chi_{(-b,b)}(\cdot)\right)(\xi) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(b\xi)}{\xi},$$

yields

$$\mathcal{F}\left(\sqrt{\frac{\pi}{2}}\,\chi_{(\tau-t,t-\tau)}(\cdot)\right)(\xi) = \frac{\sin(\xi(t-\tau))}{\xi} .$$

Substituting this result into (14), using the convolution property of Fourier transforms, and interchanging the order of integration in a double integral gives

$$\mathcal{F}(u(\cdot,t))(\xi) = \int_0^t \mathcal{F}(f(\cdot,\tau))(\xi)\,\mathcal{F}\left(\sqrt{\frac{\pi}{2}}\,\chi_{(\tau-t,t-\tau)}(\cdot)\right)(\xi)\,d\tau$$

$$= \int_0^t \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(f(\cdot,\tau) * \sqrt{\frac{\pi}{2}}\,\chi_{(\tau-t,t-\tau)}(\cdot)\right)(\xi)\,d\tau$$

$$= \mathcal{F}\left(\frac{1}{2}\int_0^t f(\cdot,\tau) * \chi_{(\tau-t,t-\tau)}(\cdot)\,d\tau\right)(\xi).$$

It follows from the uniqueness theorem that

$$u(x,t) = \frac{1}{2}\int_0^t \left[f(\cdot,\tau) * \chi_{(\tau-t,t-\tau)}(\cdot)\right](x)\,d\tau, \quad (-\infty<x<\infty, 0<t<\infty)$$

Writing the convolution product as an integral, this is equivalent to

$$u(x,t) = \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} f(y,\tau)\,\chi_{(\tau-t,t-\tau)}(x-y)\,dy\,d\tau$$

or $$\boxed{u(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+t-\tau} f(y,\tau)\,dy\,d\tau} \quad (-\infty<x<\infty, 0<t<\infty).$$

#6. $\begin{cases} u_{tt} - u_{xx} + 2u_t \overset{①}{=} 0 & \text{if } 0 < x < \pi,\ 0 < t < \infty, \\ u_x(0,t) \overset{②}{=} 0 \overset{③}{=} u_x(\pi,t) & \text{if } t \geq 0, \\ u(x,0) \overset{④}{=} 0 \text{ and } u_t(x,0) \overset{⑤}{=} x^2 & \text{if } 0 \leq x \leq \pi, \end{cases}$

(20) (a) We seek nontrivial solutions of ①-②-③-④ of the form $u(x,t) = X(x)T(t)$. Substituting this functional form in ① yields $X(x)T''(t) - X''(x)T(t) + 2X(x)T'(t) = 0$ so $\frac{T''(t)}{T(t)} + \frac{2T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda$. Hence, applying ②,③, and ④ gives

$$\begin{cases} X''(x) + \lambda X(x) \overset{⑥}{=} 0, \quad X'(0) \overset{⑦}{=} 0 \overset{⑧}{=} X'(\pi), \\ T''(t) + 2T'(t) + \lambda T(t) \overset{⑨}{=} 0, \quad T(0) \overset{⑩}{=} 0. \end{cases}$$

Using the facts about eigenvalue problems with Neumann B.C.s from Sec. 4.2, the eigenvalues of ⑥-⑦-⑧ are $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 = n^2$ and eigenfunctions $X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right) = \cos(nx)$ $(n = 0, 1, 2, \ldots)$. Substituting $\lambda = \lambda_n = n^2$ into ⑨-⑩ leads to $T_n''(t) + 2T_n'(t) + n^2 T_n(t) \overset{⑪}{=} 0$, $T_n(0) \overset{⑫}{=} 0$. Assuming $T_n(t) = e^{rt}$ in ⑩ yields $r^2 + 2r + n^2 = 0$ so $r = \frac{-2 \pm \sqrt{4 - 4n^2}}{2} = -1 \pm \sqrt{1 - n^2}$.

If $n = 0$ then $r = -1 \pm 1$ so $r = 0$ or $r = -2$ and $T_0(t) = c_1 + c_2 e^{-2t}$. Then ⑫ implies $c_1 + c_2 = 0$ so $T_0(t) = 1 - e^{-2t}$ (up to a constant factor).

If $n = 1$ then $r = -1$ (multiplicity 2) so $T_1(t) = c_1 e^{-t} + c_2 t e^{-t}$. Then ⑫ implies $c_1 = 0$ so $T_1(t) = te^{-t}$ (up to a constant factor).

If $n \geq 2$, then $r = -1 \pm i\sqrt{n^2 - 1}$ so $T_n(t) = e^{-t}\left[c_1 \cos(t\sqrt{n^2-1}) + c_2 \sin(t\sqrt{n^2-1})\right]$. Then ⑫ implies $c_1 = 0$ so $T_n(t) = e^{-t}\sin(t\sqrt{n^2-1})$ (up to a constant factor).

Therefore

$$u_0(x,t) = X_0(x)T_0(t) = 1 - e^{-2t}$$

$$u_1(x,t) = X_1(x)T_1(t) = \cos(x)te^{-t}$$

$$u_n(x,t) = X_n(x)T_n(t) = \cos(nx)e^{-t}\sin(t\sqrt{n^2-1}) \quad \text{if } n = 2, 3, 4, \ldots$$

Consequently, a formal solution to ①-②-③-④ is

$$u(x,t) \overset{(13)}{=} \sum_{n=0}^{\infty} A_n u_n(x,t) = A_0(1-e^{-2t}) + A_1\cos(x)te^{-t} + \sum_{n=2}^{\infty} A_n \cos(nx) e^{-t} \sin(t\sqrt{n^2-1})$$

where $A_0, A_1, A_2, \dots$ are "arbitrary" constants. To apply ⑤, we need

$$u_t(x,t) = 2A_0 e^{-2t} + A_1\cos(x)(1-t)e^{-t} + \sum_{n=2}^{\infty} A_n \cos(nx) e^{-t}\left[\sqrt{n^2-1}\cos(t\sqrt{n^2-1}) - \sin(t\sqrt{n^2-1})\right]$$

Applying ⑤ leads to

$$x^2 \overset{(14)}{=} u_t(x,0) = 2A_0 + A_1\cos(x) + \sum_{n=2}^{\infty} \sqrt{n^2-1}\, A_n \cos(nx) \qquad \text{for all } 0 \le x \le \pi.$$

From ⑭ it is apparent that we need the Fourier series of $f(x) = x^2$ with respect to the orthogonal set $\Phi = \{\cos(nx)\}_{n=0}^{\infty}$ on $[0,\pi]$. We compute the Fourier coefficients of f w.r.t. Φ as follows:

$$a_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{\int_0^\pi x^2 \cos(nx)\,dx}{\int_0^\pi \cos^2(nx)\,dx} = \begin{cases} \frac{1}{\pi}\int_0^\pi x^2\,dx & \text{if } n=0, \\ \frac{2}{\pi}\int_0^\pi x^2\cos(nx)\,dx & \text{if } n=1,2,3,\dots \end{cases}$$

Then $\frac{1}{\pi}\int_0^\pi x^2\,dx = \left.\frac{x^3}{3\pi}\right|_0^\pi = \frac{\pi^2}{3}$

and if $n \ge 1$,

$$\frac{2}{\pi}\int_0^\pi \underbrace{x^2}_{u}\underbrace{\cos(nx)\,dx}_{dv} = \frac{2}{\pi}\left(\cancel{\left.\frac{x^2\sin(nx)}{n}\right|_0^\pi}^{\,0} - \int_0^\pi \overbrace{2x}^{u}\overbrace{\frac{\sin(nx)}{n}\,dx}^{dv} \right) = \frac{2}{\pi}\left(\left.2x\frac{\cos(nx)}{n^2}\right|_0^\pi - \cancel{\int_0^\pi 2\frac{\cos(nx)}{n^2}\,dx}^{\,0} \right) = \frac{4(-1)^n}{n^2}.$$

Standard convergence theorems (Theorem 4^∞ in Sec. 5.4) show that the Fourier series of $f(x) = x^2$ converges to the function value at each x in $[0,\pi]$:

$$x^2 \overset{(15)}{=} \sum_{n=0}^{\infty} a_n \cos(nx) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx) \qquad \text{for all } 0 \le x \le \pi.$$

Comparing ⑭ and ⑮ we see that $2A_0 = \frac{\pi^2}{3}$, $A_1 = -4$, and $\sqrt{n^2-1}\,A_n = \frac{4(-1)^n}{n^2}$ if $n \ge 2$.

Consequently, substituting for the coefficients in ⑬ gives a solution to ①-②-③-④-⑤:

$$\boxed{u(x,t) = \frac{\pi^2}{6}(1-e^{-2t}) - 4\cos(x)te^{-t} + \sum_{n=2}^{\infty} \frac{4(-1)^n}{n^2\sqrt{n^2-1}}\cos(nx)e^{-t}\sin(t\sqrt{n^2-1})} \quad \text{if } 0 \le x \le \pi,\ 0 \le t < \infty.$$

#6 (cont.)

(4) (b) Let $u=u(x,t)$ be a C^2-solution to ①-②-③ and consider its energy function $E(t)=\int_0^{\pi}\left(\frac{1}{2}u_t^2(x,t)+\frac{1}{2}u_x^2(x,t)\right)dx$. Then $\frac{dE}{dt}=\frac{d}{dt}\left(\int_0^{\pi}\left(\frac{1}{2}u_t^2(x,t)+\frac{1}{2}u_x^2(x,t)\right)dx\right)$

$$=\int_0^{\pi}\frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2(x,t)+\frac{1}{2}u_x^2(x,t)\right)dx=\int_0^{\pi}\Big[u_t(x,t)u_{tt}(x,t)+\underbrace{u_x(x,t)}_{U}\,\overbrace{u_{xt}(x,t)}^{dV}\Big]dx.$$

Integrating the second term in the integrand by parts and applying the B.C.s ②-③ gives

$$\frac{dE}{dt}=\int_0^{\pi}u_t(x,t)u_{tt}(x,t)\,dx+\cancelto{0}{u_x(x,t)u_t(x,t)\Big|_{x=0}^{\pi}}-\int_0^{\pi}u_t(x,t)u_{xx}(x,t)\,dx$$

$$=\int_0^{\pi}u_t(x,t)\big(u_{tt}(x,t)-u_{xx}(x,t)\big)dx$$

Substituting from ① then yields $\frac{dE}{dt}=\int_0^{\pi}u_t(x,t)\left[-2u_t(x,t)\right]dx=-2\int_0^{\pi}u_t^2(x,t)\,dx\le 0$.

That is, the energy function of u is a decreasing function on the interval $t\ge 0$.

(5) (c) Suppose $u=v(x,t)$ were another solution to ①-②-③-④-⑤ and let $w(x,t)=u(x,t)-v(x,t)$ where $u=u(x,t)$ is the solution found in part (a). Then w solves the system

$$\begin{array}{lll} ①' & w_{tt}-w_{xx}+2w_t=0 & \text{if } 0<x<\pi,\ 0<t<\infty, \\ ②'-③' & w_x(0,t)=0=w_x(\pi,t) & \text{if } t\ge 0, \\ ④'-⑤' & w(x,0)=0=w_t(x,0) & \text{if } 0\le x\le\pi. \end{array}$$

By part (b) the energy function of w, $E(t)=\int_0^{\pi}\left(\frac{1}{2}w_t^2(x,t)+\frac{1}{2}w_x^2(x,t)\right)dx$, is a decreasing function on $t\ge 0$. Hence $0\le E(t)\le E(0)$ for all $t\ge 0$. But ④′ and ⑤′ imply $E(0)=\int_0^{\pi}\left[\frac{1}{2}w_t^2(x,0)+\frac{1}{2}w_x^2(x,0)\right]dx=0$ and the vanishing theorem shows $w_t(x,t)=0=w_x(x,t)$ if $0\le x\le\pi,\ 0\le t<\infty$. Thus $w(x,t)$ is constant in the strip $0\le x\le\pi,\ 0\le t<\infty$. But ④′ implies the constant is zero. Therefore $v(x,t)=u(x,t)$ for all $0\le x\le\pi,\ 0\le t<\infty$. That is, the solution to ①-②-③-④-⑤ is unique.

#7.
$$\begin{cases} u_{xx}+u_{yy}+u_{zz} \overset{①}{=} 0 & \text{if } 0<x<1,\ 0<y<1,\ 0<z<1, \\ u_x(0,y,z) \overset{②}{=} 0 \overset{③}{=} u_x(1,y,z) & \text{if } 0\le y\le 1,\ 0\le z\le 1, \\ u_z(x,y,0) \overset{④}{=} 0 \overset{⑤}{=} u_z(x,y,1) & \text{if } 0\le x\le 1,\ 0\le y\le 1, \\ u_y(x,0,z) \overset{⑥}{=} 0 \text{ and } u(x,1,z) \overset{⑦}{=} 4\sin^2(\pi x)\sin^2(\pi z) & \text{if } 0\le x\le 1,\ 0\le z\le 1, \end{cases}$$

is the system satisfied by a solution to problem 7. We apply the method of separation of variables. We seek nontrivial solutions of the form $u(x,y,z) = X(x)Y(y)Z(z)$ to the homogeneous portion of the problem: ①-②-③-④-⑤-⑥. Substituting this functional form into ① yields

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0.$$

Dividing by $X(x)Y(y)Z(z)$ and rearranging, we have

$$\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\frac{X''(x)}{X(x)}.$$

For fixed y and z, if we vary the independent variable x over the interval $[0,1]$ then we see that $-X''(x)/X(x)$ must be constant, say

$$\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\frac{X''(x)}{X(x)} = \text{constant} = \lambda.$$

Rearranging the left and right members of the string of equalities above and repeating the argument, we find

$$\frac{Y''(y)}{Y(y)} - \lambda = -\frac{Z''(z)}{Z(z)} = \text{constant} = \mu.$$

Therefore we arrive at the following system of coupled ODEs:

$$X''(x) + \lambda X(x) = 0 \quad \text{if } 0<x<1,$$

$$Z''(z) + \mu Z(z) = 0 \quad \text{if } 0<z<1,$$

$$Y''(y) - (\lambda+\mu) Y(y) = 0 \quad \text{if } 0<y<1.$$

Applying B.C. (2) to a nontrivial solution to the homogeneous portion of the problem of the form $u(x,y,z) = X(x)Y(y)Z(z)$ we find

$$X'(0)Y(y)Z(z) = 0 \quad \text{if} \quad 0 \le y \le 1, 0 \le z \le 1.$$

If $X'(0) \neq 0$ then we would have $u(x,y,z) = X(x)Y(y)Z(z) = 0$ for all (x,y,z) in the unit cube, in contradiction to nontriviality of u. Therefore we must have $X'(0) = 0$. Arguing in a similar way using (3)-(4)-(5)-(6), we obtain

$$X''(x) + \lambda X(x) \overset{(7)}{=} 0, \quad X'(0) \overset{(8)}{=} 0 \overset{(9)}{=} X'(1),$$

$$Z''(z) + \mu Z(z) \overset{(10)}{=} 0, \quad Z'(0) \overset{(11)}{=} 0 \overset{(12)}{=} Z'(1),$$

$$Y''(y) - (\lambda + \mu) Y(y) \overset{(13)}{=} 0, \quad Y'(0) \overset{(14)}{=} 0.$$

Using the results of Sec. 4.2, we see that the eigenvalues and eigenfunctions for the Neumann problems (7)-(8)-(9) and (10)-(11)-(12) are, respectively,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \overset{(15)}{=} n^2\pi^2, \quad X_n(x) = \cos\left(\frac{n\pi x}{l}\right) \overset{(16)}{=} \cos(n\pi x), \quad (n = 0, 1, 2, \ldots),$$

$$\mu_m = \left(\frac{m\pi}{l}\right)^2 \overset{(17)}{=} m^2\pi^2, \quad Z_m(z) = \cos\left(\frac{m\pi z}{l}\right) \overset{(18)}{=} \cos(m\pi z), \quad (m = 0, 1, 2, \ldots).$$

Substituting $\lambda = \lambda_n = n^2\pi^2$ and $\mu = \mu_m = m^2\pi^2$ into (13)-(14) yields

$$Y''_{n,m}(y) - (n^2 + m^2)\pi^2 Y_{n,m}(y) \overset{(13')}{=} 0, \quad Y'_{n,m}(0) \overset{(14')}{=} 0.$$

The general solution of (13') is

$$Y_{n,m}(y) = \begin{cases} c_1 + c_2 y & \text{if } (n,m) = (0,0) \\ c_1 \cosh\left(\pi y \sqrt{m^2 + n^2}\right) + c_2 \sinh\left(\pi y \sqrt{m^2 + n^2}\right) & \text{if } (n,m) \neq (0,0) \end{cases}$$

Applying (14') leads to $c_2 = 0$. Therefore, up to a constant factor,

$$Y_{n,m}(y) \overset{(19)}{=} \begin{cases} 1 & \text{if } (n,m) = (0,0), \\ \cosh\left(\pi y \sqrt{m^2 + n^2}\right) & \text{if } (n,m) \neq (0,0). \end{cases}$$

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$$\text{Thus,}\quad u_{n,m}(x,y,z) = \begin{cases} 1 & \text{if } (n,m) = (0,0), \\ \cos(n\pi x)\cos(m\pi z)\cosh(\pi y\sqrt{m^2+n^2}) & \text{if } (n,m) \neq (0,0). \end{cases}$$

Consequently, the superposition principle implies that a formal solution to ①-②-③-④-⑤-⑥ is

$$u(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\, u_{n,m}(x,y,z),$$

$$u(x,y,z) \overset{(20)}{=} c_{0,0} + \sum_{\substack{n=0 \\ (n,m)\neq(0,0)}}^{\infty}\sum_{m=0}^{\infty} c_{n,m} \cos(n\pi x)\cos(m\pi z)\cosh(\pi y\sqrt{m^2+n^2})$$

where $c_{n,m}$ $(n = 0,1,2,\dots;\ m = 0,1,2,\dots)$ are arbitrary constants. We want to choose these constants such that the nonhomogeneous B.C. ⑦ is satisfied:

$$4\sin^2(\pi x)\sin^2(\pi z) = u(x,1,z) \overset{(21)}{=} c_{0,0} + \sum_{\substack{n=0 \\ (n,m)\neq(0,0)}}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\cosh(\pi\sqrt{m^2+n^2})\cos(n\pi x)\cos(m\pi z)$$

for all $0 \le x \le 1$, $0 \le z \le 1$. Applying the given trig identity we find

$$4\sin^2(\pi x)\sin^2(\pi z) = 4\left(\frac{1-\cos(2\pi x)}{2}\right)\left(\frac{1-\cos(2\pi z)}{2}\right)$$

$$\overset{(22)}{=} 1 - \cos(2\pi x) - \cos(2\pi z) + \cos(2\pi x)\cos(2\pi z)$$

Comparing (21) and (22), we see that

$$c_{0,0} = 1,\quad \cosh(2\pi)\, c_{2,0} = -1,\quad \cosh(2\pi)\, c_{0,2} = -1,\quad \cosh(\pi\sqrt{8})\, c_{2,2} = 1,$$

and all other $c_{n,m} = 0$, is a choice which satisfies ⑦. That is, substituting these coefficients into (20), we find

(25)

$$\boxed{\begin{aligned} u(x,y,z) = 1 &- \frac{1}{\cosh(2\pi)}\cos(2\pi x)\cosh(2\pi y) - \frac{1}{\cosh(2\pi)}\cos(2\pi z)\cosh(2\pi y) \\ &+ \frac{1}{\cosh(2\pi\sqrt{2})}\cos(2\pi x)\cos(2\pi z)\cosh(2\pi\sqrt{2}\, y)\end{aligned}}$$

solves ①-②-③-④-⑤-⑥-⑦.

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(15) **Bonus:** Yes, the solution to problem 7 is unique. To see this, suppose $u = v(x,y,z)$ is another solution to problem 7 and let

$$w(x,y,z) = u(x,y,z) - v(x,y,z)$$

where $u = u(x,y,z)$ is the solution we found to problem 7. Then w solves

$$\nabla^2 w \overset{①}{=} 0 \quad \text{in} \quad C: 0<x<1,\ 0<y<1,\ 0<z<1,$$

$$w(x,1,z) \overset{②}{=} 0 \quad \text{for} \quad 0 \le x \le 1,\ 0 \le z \le 1,$$

and $\dfrac{\partial w}{\partial n} \overset{③}{=} \nabla w \cdot \vec{n} = 0$ on the other five faces of C.

Consider the energy,

$$E = \frac{1}{2}\iiint_C |\nabla w|^2\, dV = \frac{1}{2}\iiint_C (w_x^2 + w_y^2 + w_z^2)\, dV$$

of the solution w. We apply Green's first identity,

$$\iint_{\partial D} f\frac{\partial g}{\partial n}\, dS = \iiint_D \nabla f \cdot \nabla g\, dV + \iiint_D f\nabla^2 g\, dV,$$

(8)

(8)

with $f = g = w$ and $D = C$ to obtain

$$\iint_{\partial C} w \frac{\partial w}{\partial n}\, dS = \iiint_{C} |\nabla w|^2\, dV + \iiint_{C} w \nabla^2 w\, dV.$$

But ① implies $\iiint_{C} w \nabla^2 w\, dV = 0$ and ② and ③ imply $\iint_{\partial C} w \frac{\partial w}{\partial n}\, dS = 0$.

Therefore

$$2E = \iiint_{C} |\nabla w|^2 dV = 0.$$

The vanishing theorem then implies $0 = w_x = w_y = w_z$ in C so $w(x,y,z) =$ constant in $\overline{C}$. But ② then implies this constant is zero. That is, $v(x,y,z) = u(x,y,z)$ in $\overline{C}$ so the solution we found in problem 7 is unique.

(15)

Math 5325

Final Exam

Summer 2015

mean: 132.7

median: 142

standard deviation: 41.6

number of exams: 15

Distribution of Scores:

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
174-200	A	A	3
146-173	B	B	4
120-145	C	B	2
100-119	C	C	1
0-99	F	D	5