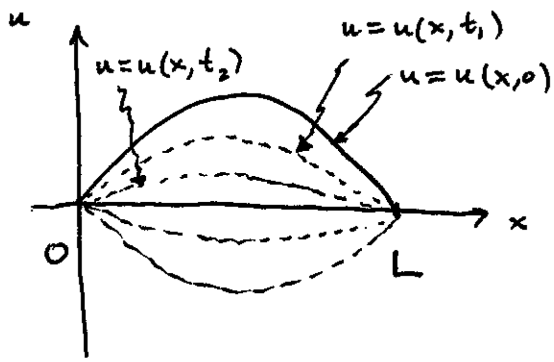
Chap. 1 Where PDE's Come From

Sec. 1.1 What is a PDE?

(Use a rubber band prop.) An elastic string is stretched to a length L and fixed at its endpoints. The string is distorted and then released at a certain instant, say $t=0$. We seek the transverse displacement $u(x,t)$ of the string at position x in $[0,L]$ and time $t \geq 0$.



We will see in Sec. 1.3 that for "small" displacements, the equation governing the motion is (approximately)

$$(*) \qquad \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{(1-D Wave Equation)}$$

where c is a positive constant that depends on the physical properties (tension and density) of the string.

[Need to informally define "PDE" and "solution" of a PDE here.]

(over for formal)

Ex 1 (a) Verify that for each $n = 1, 2, 3, \ldots$ the function

$$u_n(x,t) = \cos\left(\frac{n\pi c t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

is a solution of (*) in the xt-plane: $-\infty < t < \infty$, $-\infty < x < \infty$.

(b) Verify that if f and g are any twice-differentiable functions of a single real variable then

$$u(x,t) = f(x+ct) + g(x-ct)$$

Much discussion! (Traveling waves)

is a solution of (*) in the xt-plane.

Notes: ① Solutions of the form (a) satisfy the initial/boundary conditions

(B.C.) $\quad 0 = u(0,t) = u(L,t)$ for all $t \geq 0$.

(I.C.) $\quad \frac{\partial u}{\partial t}(x,0) = 0$ for all $0 \leq x \leq L$.

Solutions of the form (b) need not satisfy (B.C.) and (I.C.)

To HERE (Day 1)

② Solutions of the form (b) involve two "arbitrary" functions. In modeling physical phenomena, the PDE governing the evolution of the system must be supplemented with appropriate initial/boundary conditions to identify the "physically relevant" solutions among the vast number of possible solutions.

③ The wave operator $\mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}$ in (*) is second-order and linear; i.e. $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$ and $\mathcal{L}(ku) = k\mathcal{L}(u)$. [Here $u = u(x,t)$ and $v = v(x,t)$ are arbitrary twice-differentiable functions of x and t and k is an arbitrary constant.] Therefore (*) is a linear homogeneous PDE: $\mathcal{L}(u) = 0$. A linear nonhomogeneous PDE has the form

$$\mathcal{L}(u) = f(x,t)$$

where $\mathcal{L}$ is a linear PD operator and $f = f(x,t)$ is a specified nonzero function

Superposition Principle: If $u_1, u_2, \ldots, u_n$ are ~~an infinite number of~~ finitely many solutions to a linear homogeneous PDE $\mathcal{L}(u) = 0$ and $k_1, k_2, \ldots, k_n$ are any constants then the linear combination of solutions

$$u(x,t) = k_1 u_1(x,t) + k_2 u_2(x,t) + \cdots + k_n u_n(x,t)$$

is also a solution to $\mathcal{L}(u) = 0$.

For example, $$u(x,t) = \sum_{j=1}^{n} k_j \cos\left(\frac{j\pi ct}{L}\right)\sin\left(\frac{j\pi x}{L}\right)$$

$$= k_1 \cos\left(\frac{\pi ct}{L}\right)\sin\left(\frac{\pi x}{L}\right) + \cdots + k_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

solves $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ for any integer $n \geq 1$ and any constants $k_1, \ldots, k_n$.

Ex 2| Determine the order and whether or not the dispersive wave equation

(3rd order) → $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \longleftarrow \left(u = \frac{x}{t} \text{ is a solution on } 0 < t < \infty,\ -\infty < x < \infty \text{ but } 2u \text{ is not.} \right)$$

is linear. (Korteweg-deVries equation; cf. Sec 14.2, pp. 367–374.)

Ex 3| (#10, p.5) Show that the solutions of the ODE

(†) $$u''' - 3u'' + 4u = 0$$

form a vector space. Find a basis for the solution space of (†).

By IV and FACT I on p.2 of "Vector Spaces" handout, it suffices to show that if $\mathcal{L}(u_1) = 0$ and $\mathcal{L}(u_2) = 0$ then $\mathcal{L}(c_1 u_1 + c_2 u_2) = 0$ for any constants c_1 and c_2. (Check this)
$\{e^{-t}, e^{2t}, te^{2t}\}$ is a basis. Note: $0 = m^3 - 3m^2 + 4 = (m+1)(m^2 - 4m + 4) = (m+1)(m-2)^2$.