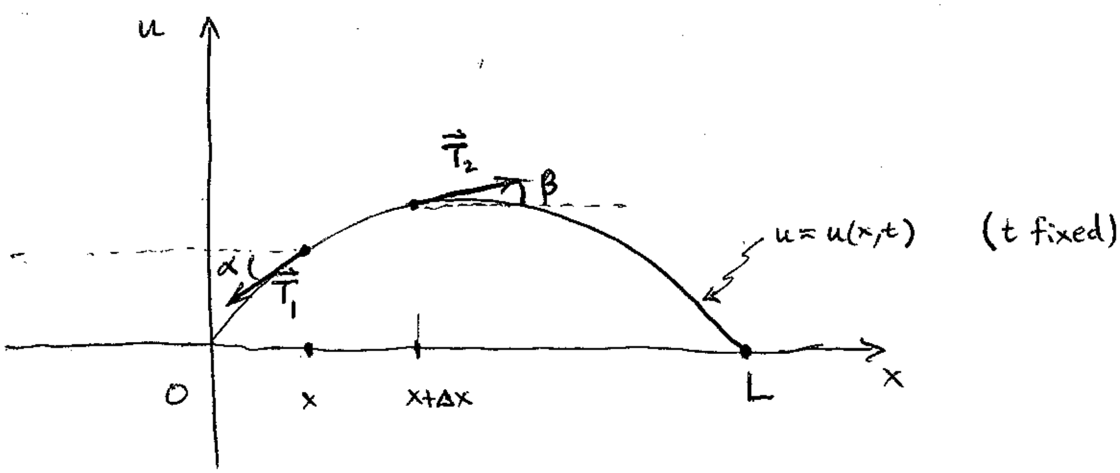
Derivation of the One-Dimensional Wave Equation

Problem: An elastic string is stretched to a length L and fixed at its endpoints. If the string is distorted and then released at time $t = 0$, find an equation governing the transverse displacement $u = u(x,t)$ of the string at (horizontal) position x and time t.

Solution: Reality is infinitely more complex than any mathematical model can describe. Consequently, we will make the following modeling assumptions so as to distill out the "essential" features of the problem. (For a more detailed treatment of the equation governing vibratory motion of an elastic string, see the references at the end of this handout.)

1. The string is so thin that its cross sections move as single points.
2. The string will be represented as a continuum with linear mass density $\rho(x)$ at position x.
3. The string performs only small transverse vibrations in a vertical plane.
4. The string is perfectly flexible; i.e. tensile forces are transmitted tangentially at all points along the string.
5. The dissipative forces retarding the string's motion are small and may be safely neglected.
6. The tensions exerted at the endpoints of the string, to hold it motionless there, are the only external forces acting on the string.

Fix a time $t > 0$ and a horizontal position x between 0 and L. Consider the segment of string between x and $x + \Delta x$. By assumption 4, the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ that act on the endpoints of this segment are tangential to the segment there. (See the diagram below.)



Let α and β, respectively, denote the angles that $\mathbf{T}_1$ and $\mathbf{T}_2$ make with respect to horizontal. Then

(1)
$$\tan(\alpha) = \text{slope of tangent line to string at } x \text{ (for fixed } t\text{)} = u_x(x,t)$$

and it follows that

(2)
$$\cos(\alpha) = \frac{1}{\sqrt{1+u_x^2(x,t)}} \quad \text{and} \quad \sin(\alpha) = \frac{u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}$$

Similarly,

(3)
$$\cos(\beta) = \frac{1}{\sqrt{1+u_x^2(x+\Delta x,t)}} \quad \text{and} \quad \sin(\beta) = \frac{u_x(x+\Delta x,t)}{\sqrt{1+u_x^2(x+\Delta x,t)}}$$

Applying Newton's second law of motion, $\mathbf{F} = m\mathbf{a}$, to the segment of string between x and $x + \Delta x$, and resolving this vector equation into horizontal and vertical components yields

(4) $$|\mathbf{T}_2|\cos(\beta) - |\mathbf{T}_1|\cos(\alpha) = 0 \qquad \text{(by assumption 3)}$$

(5) $$|\mathbf{T}_2|\sin(\beta) - |\mathbf{T}_1|\sin(\alpha) = \int_x^{x+\Delta x} \underbrace{u_{tt}(\xi, t)}_{\text{acceleration}} \underbrace{\rho(\xi)\,d\xi}_{\text{element of mass}}$$

Substituting from (2) and (3) into (4) yields

(6) $$|\mathbf{T}_2| = |\vec{T}_1| \frac{\cos(\alpha)}{\cos(\beta)} = |\vec{T}_1| \frac{\sqrt{1 + u_x^2(x+\Delta x, t)}}{\sqrt{1 + u_x^2(x,t)}} .$$

Substituting (6), (2), and (3) into (5) produces

(7) $$|\vec{T}_1| \left(\frac{u_x(x+\Delta x, t)}{\sqrt{1 + u_x^2(x,t)}} - \frac{u_x(x,t)}{\sqrt{1 + u_x^2(x,t)}} \right) = \int_x^{x+\Delta x} \rho(\xi) u_{tt}(\xi, t)\, d\xi .$$

Rearranging and dividing both sides of (7) by Δx results in

(8) $$\frac{|\vec{T}(x, u, u_x, t)|}{\sqrt{1 + u_x^2(x,t)}} \left[\frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x} \right] = \frac{1}{\Delta x} \int_x^{x+\Delta x} \rho(\xi) u_{tt}(\xi, t)\, d\xi .$$

Now, taking the limit in (8) as $\Delta x \to 0$ and using the continuity of the integrand in the right member gives

(9) $$\frac{|\vec{T}(x, u, u_x, t)|}{\sqrt{1 + u_x^2(x,t)}} u_{xx}(x,t) = \rho(x) u_{tt}(x,t)$$

or equivalently,

(10) $$u_{tt}(x,t) - \frac{|\vec{T}(x, u, u_x, t)|}{\rho(x)\sqrt{1 + u_x^2(x,t)}} u_{xx}(x,t) = 0 .$$

For small vibrations, u and its derivatives are very much less than 1. Consequently, $\sqrt{1 + u_x^2(x,t)}$ is approximately 1, and it follows from (2), (3), and (4) that $|\mathbf{T}(x, u, u_x, t)|$ is approximately constant, say T_0. Thus for small vibrations, (10) is approximately

(11) $$u_{tt}(x,t) - \frac{T_0}{\rho(x)} u_{xx}(x,t) = 0 .$$

When the string is homogeneous, i.e. when $\rho(x) =$ constant $= \rho_0$ for all x between 0 and L, (11) reduces to

(12) $$\boxed{u_{tt} - c^2 u_{xx} = 0}$$

where $c = \sqrt{T_0/\rho_0}$. Equation (12) is known as the one-dimensional (undamped) wave equation.

References.
~~Remarks. For a more detailed treatment of the equation governing vibratory motion of an elastic string, see the following sources~~.

1. *A First Course in Partial Differential Equations* by Hans Weinberger, Wiley, 1965, pp. 1-5.
2. S. Antman, *The Equations for Large Vibrations of Strings*, American Mathematical Monthly 87 (May 1980), pp. 359-370.

2-D wave equation: $$\frac{\partial^2 u}{\partial t^2} - c^2\overbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}^{\nabla^2 u \quad \text{2-D Laplacian}} = 0$$

3-D wave equation: $$\frac{\partial^2 u}{\partial t^2} - c^2\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\nabla^2 u \quad \text{3-D Laplacian}} = 0$$

The wave equation: $$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{(hyperbolic)}$$

The heat (or diffusion) equation: $$\frac{\partial u}{\partial t} - k\nabla^2 u = 0 \qquad \text{(parabolic)}$$

Laplace's equation: $$\nabla^2 u = 0 \qquad \text{(elliptic)}$$

Motivate with "standing waves"
i.e. waves $u = u(\vec{x})$ that are independent of time t.

Derive the 3-D heat (or diffusion) equation now. (See next page.)

Derivation of the Heat (or Diffusion) Equation

Problem: A solid body occupies a region $\mathcal{R}$ in space. If $\mathcal{R}$ contains no sources or sinks of heat, find an equation governing the temperature $u(x,y,z,t)$ of the body at position (x,y,z) in $\mathcal{R}$ and time $t>0$.

Solution: It is an empirical physical "law" that heat energy flows in a solid body in the direction of decreasing temperature, and evidence shows that the rate of heat flow (per unit time) is approximately proportional to the magnitude of the temperature gradient. Thus the velocity of heat flow in $\mathcal{R}$ is

(1) $$\mathbf{v} = -k\nabla u$$

where $k = k(x,y,z)$ is the thermal conductivity of the material at position (x,y,z) in $\mathcal{R}$.

Let C be any cube contained in $\mathcal{R}$. The heat leaving C per unit time is

(2) $$-\frac{dH}{dt} = \iint_{\partial C} \vec{v}\cdot\vec{n}\, dS$$

where $\mathbf{v}\cdot\mathbf{n}$ is the component of $\mathbf{v}$ in the direction of the outward-pointing unit normal $\mathbf{n}$ to the boundary ∂C of C, and dS denotes the element of surface area on ∂C. From (1) and the divergence theorem (Appendix A.3, p. 393) we obtain

(3) $$\iint_{\partial C} \vec{v}\cdot\vec{n}\, dS = \iiint_C \nabla\cdot\vec{v}\, dV \overset{(1)}{=} -\iiint_C \nabla\cdot(k\nabla u)\, dV.$$

On the other hand, the total amount of heat in C at time t is

(4) $$H(t) = \iiint_C E(x,y,z,u(x,y,z,t))\, dV$$

where $E(x,y,z,u)$ denotes the energy density (i.e. energy per unit volume) at position (x,y,z) and temperature u. Hence the time rate of change of H in C is

(5) $$\begin{aligned}\frac{dH}{dt} &= \frac{d}{dt}\iiint_C E(x,y,z,u(x,y,z,t))\, dV \\ &\overset{\text{Thm.4, p.391}}{=} \iiint_C \frac{\partial}{\partial t}\Big(E(x,y,z,u(x,y,z,t))\Big)\, dV \\ &= \iiint_C \frac{\partial E}{\partial u}\frac{\partial u}{\partial t}\, dV,\end{aligned}$$

and this must be equal to the amount of heat leaving C. From (3) and (5) we thus have

(6) $$-\iiint_C \nabla\cdot(k\nabla u)\, dV = -\frac{dH}{dt} = -\iiint_C \frac{\partial E}{\partial u}\frac{\partial u}{\partial t}\, dV$$

or equivalently,

(7) $$\iiint_C \left[\frac{\partial E}{\partial u}\frac{\partial u}{\partial t} - \nabla\cdot(k\nabla u)\right] dV = 0 .$$

Since this relation holds for every cube C contained in $\mathcal{R}$ and the integrand is **assumed** to be continuous in $\mathcal{R}$, the second vanishing theorem (Appendix A.1, p. 386) implies that the integrand must be zero everywhere. That is,

(8) $$\frac{\partial E}{\partial u}\frac{\partial u}{\partial t} - \nabla\cdot(k\nabla u) = 0 \quad \text{in } \mathcal{R}.$$

If the material in $\mathcal{R}$ is homogeneous, then its thermal conductivity is constant:

(9) $$k(x,y,z) = \text{constant} = k_0$$

Furthermore, for moderate temperature ranges and most commonly occurring materials, the rate of change of energy density with temperature is nearly constant:

(10) $$\frac{\partial E}{\partial u} \cong \text{constant} = \sigma\rho$$

where σ denotes the specific heat (i.e. rate of change of heat energy per unit mass per unit temperature) and ρ denotes the mass density (i.e. mass per unit volume). Substituting from (9) and (10) into (8) yields the classical three-dimensional heat (or diffusion) equation

(11) $$\boxed{u_t - c^2\nabla^2 u = 0}$$

where $c = \sqrt{\dfrac{k_0}{\sigma\rho}}$.

2-D wave eqn. $$\frac{\partial^2 u}{\partial t^2} - c^2\overbrace{\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)}^{\nabla^2 u \text{ (2-D Laplacian of } u)} = 0$$

3-D " " $$\frac{\partial^2 u}{\partial t^2} - c^2\underbrace{\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)}_{\nabla^2 u \text{ (3-D Laplacian of } u)} = 0$$

$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0$ wave eqn (hyperbolic) ← To here Day 4

Derive 3-D heat equation here.

$\frac{\partial u}{\partial t} - k\nabla^2 u = 0$ heat (or diffusion) eqn. (parabolic)

$\nabla^2 u = 0$ Laplace's eqn (elliptic)

see next page

Ex 2 (3-D heat equation) ~~Read carefully text's derivation. Responsible for derivation of 3-D heat equation on Exam I~~

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$$

Ex 3 (#6, p. 19) $\frac{\partial u}{\partial t} - k\nabla^2 u = 0$

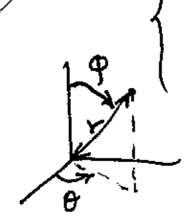
In cylindrical coordinates $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$.

If $u = u(r,t)$, independent of θ and z, then $\frac{\partial u}{\partial \theta} = \frac{\partial^2 u}{\partial \theta^2} = 0 = \frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial z^2}$

so the heat eqn becomes

$$\frac{\partial u}{\partial t} - k\overbrace{\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)}^{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)} = 0.$$

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$



Give Hint on #7: In spherical coordinates

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial u}{\partial \varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}.$$

To here Day 5 (Spent 15 min. at beginning of period working HW problem)