Sec. 1.4 Initial and Boundary Conditions

Mathematical model for small vibrations of the undamped vibrating string with endpoints fixed:

PDE: $u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0$ for $0 < x < L$ and $t > 0$,

B.C.'s: $u(0,t) = 0$ and $u(L,t) = 0$ for $t \geq 0$,

I.C.'s: $u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$ for $0 \leq x \leq L$.

(Specified initial position and initial velocity functions for the initial "pluck".)

The B.C.'s above are called homogeneous Dirichlet boundary conditions. The other two commonly encountered boundary conditions are:

$u_x(0,t) = 0$ for $t \geq 0$ (homogeneous Neumann B.C.)

$u_x(0,t) + \gamma u(0,t) = 0$ for $t \geq 0$ (homogeneous Robin B.C.)

(See example on next three pages.)

Example for Initial and Boundary Conditions (Sec. 1.4)

A string of length L, linear density ρ_0, and tension T_0 has the end $x = L$ fixed on the $x-$axis. The end $x = 0$ is attached to a runner of mass m which slides in a groove on the $u-$axis under the action of a force

$$-au - bu_t + p(t)$$

in the vertical direction. (Here a and b are positive constants.)

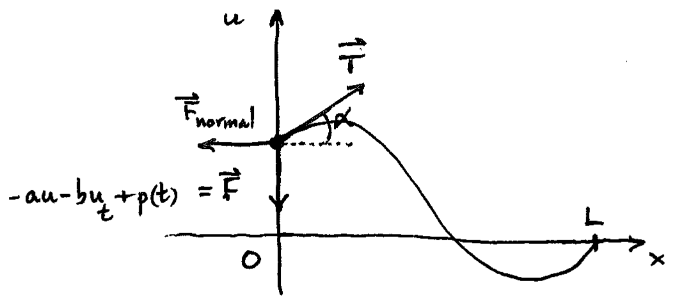
(a) Find the condition satisfied at $x = 0$ by the vertical displacement function $u = u(x,t)$ for small vibrations.

(See solution on next page.)

(b) What does this become when the mass m is negligibly small and the runner may be regarded as massless $(m = 0)$?

(c) For negligibly small mass and vibrations which are free at the left end, write the boundary condition.

(d) For negligibly small mass, when the runner is connected at the left end to a spring which exerts a restoring force proportional to the displacement, write the boundary condition.



We apply Newton's second law to the runner of mass m:

(Vertical) $\quad m u_{tt}(0,t) = -au(0,t) - bu_t(0,t) + p(t) + |\vec{T}|\sin(\alpha)$

(Horizontal) $\quad 0 = |\vec{T}|\cos(\alpha) - |\vec{F}_{normal}|$

As in the derivation of the 1-D wave equation, we have

$$\sin(\alpha) = \frac{u_x(0,t)}{\sqrt{1+\underbrace{u_x^2(0,t)}_{\text{neglect since small}}}} \approx u_x(0,t)$$

$$\cos(\alpha) = \frac{1}{\sqrt{1+\underbrace{u_x^2(0,t)}_{\text{neglect}}}} \approx 1$$

and hence $|\vec{T}| = \text{constant} = T_0$. Therefore

(a) $$\boxed{m u_{tt}(0,t) = -au(0,t) - bu_t(0,t) + p(t) + T_0 u_x(0,t)}$$

When m is negligibly small this becomes

(b) $$\boxed{0 = -au(0,t) - bu_t(0,t) + p(t) + T_0 u_x(0,t)}\ .$$

For vibrations which are free at the left end, i.e. $-au(0,t)-bu_t(0,t)+p(t) = \vec{F} = 0$, and for negligibly small mass, the boundary condition at the left end is

(c) $\boxed{0 = u_x(0,t)}$ for $t \geq 0$ (homogeneous <u>Neumann</u> B.C.)

When the runner is connected to a spring (i.e. $a = k =$ spring modulus and $b = p(t) = 0$) and its mass m is negligible, the B.C. at the left end is

$$0 = -ku(0,t) + T_0 u_x(0,t)$$

or equivalently

(d) $$\boxed{0 = u_x(0,t) + \gamma u(0,t)}$$ for $t \geq 0$ (homogeneous Robin B.C.)

where $\gamma = -\dfrac{k}{T_0}$.