Chapter 2: Waves and Diffusions

Sec 2.1 The Wave Equation

Ex 1 (Similar to #10, p.37) Solve $u_{xx} + u_{xt} - 20u_{tt} = 0$ in the xt-plane subject to the initial conditions

$$u(x,0) = 25x^2 + 64x^3 \text{ and } u_t(x,0) = -10x + 48x^2 \text{ for all } -\infty < x < \infty.$$

Soln: From previous work, (a) in Sec. 1.6 we know that the general solution to the PDE in the xt-plane is $u(x,t) = f(5x-t) + g(4x+t)$ where f and g are C^2-functions of a single real variable. We must determine f and g so the I.C.'s are satisfied.

(1) $25x^2 + 64x^3 = u(x,0) = f(5x) + g(4x)$ for all $-\infty < x < \infty$.

$$u_t(x,t) = -f'(5x-t) + g'(4x+t)$$

$$\begin{cases} (2) & -10x + 48x^2 = u_t(x,0) = -f'(5x) + g'(4x) \quad \text{for all } -\infty < x < \infty. \\ (1') & 50x + 192x^2 = 5f'(5x) + 4g'(4x) \quad \text{" " "} \end{cases}$$

Multiplying (2) by 5 and adding to (1′) gives

$$432x^2 = 9g'(4x) \implies 48x^2 = g'(4x)$$

Letting $z = 4x$ gives $g'(z) = 3z^2$

$$\implies g(z) = z^3 + c_1 \qquad \text{for all } -\infty < z < \infty$$

Multiplying (2) by −4 and adding to (1′) gives

$$90x = 9f'(5x) \implies 10x = f'(5x)$$

(Let $z = 5x$) $\implies f'(z) = 2z$

$$\implies f(z) = z^2 + c_2 \qquad \text{for all } -\infty < z < \infty$$

$$\therefore \quad u(x,t) = f(5x-t) + g(4x+t)$$

$$u(x,t) = (5x-t)^2 + c_2 + (4x+t)^3 + c_1$$

However, the constants c_1 and c_2 are not independent. For example,

$$25x^2 + 64x^3 = u(x,0) = (5x)^2 + c_2 + (4x)^3 + c_1$$

$$= 25x^2 + c_2 + 64x^3 + c_1$$

so we need $c_1 + c_2 = 0$. Therefore the particular solution to our initial value problem is

$$\boxed{u(x,t) = (5x-t)^2 + (4x+t)^3}.$$

Similar techniques give the following result:

The solution to $u_{tt} - c^2 u_{xx} = 0$ in the xt-plane satisfying the initial conditions

$$u(x,0) = \varphi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for all } -\infty < x < \infty$$

is given by

d'Alembert's formula

$$\longrightarrow \quad u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\,d\xi .$$

(Here we assume that φ is a C^2-function and ψ is a C^1-function of a single real variable.)

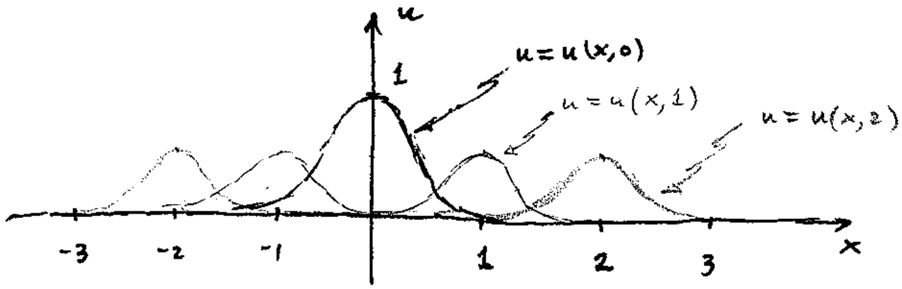
Note: You are responsible for the derivation of this result on Exam I.

Ex 2 Solve $u_{tt} - u_{xx} = 0$ in the xt-plane subject to

$$u(x,0) = e^{-x^2} \quad \text{and} \quad u_t(x,0) = 0 \quad \text{for} \quad -\infty < x < \infty .$$

Soln:
$$u(x,t) = \frac{1}{2}\left[e^{-(x-t)^2} + e^{-(x+t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t} 0\, d\xi$$

$$= \frac{1}{2}e^{-(x-t)^2} + \frac{1}{2}e^{-(x+t)^2}$$



$u = \frac{1}{2}e^{-(x-t)^2}$

Note: The solution consists of two "traveling" waves, one that moves to the right along the x-axis with speed 1 and one that moves to the left with speed 1.

$u = \frac{1}{2}e^{-(x+t)^2}$

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Ex 3 (#5, p.37 The Hammer Blow) Find the solution of $u_{tt} - c^2 u_{xx} = 0$ satisfying $u(x,0) = \varphi(x) = 0$ and $u_t(x,0) = \psi(x) = \begin{cases} 1 & \text{if } -a < x < a, \\ 0 & \text{o.w.} \end{cases}$

Sketch the profile of the string at each of the successive instants $t = \frac{a}{2c}, \frac{a}{c}, \frac{3a}{2c}, \frac{2a}{c}$, and $\frac{5a}{c}$.

Soln:
$$u(x,t) = \frac{1}{2}\left[\varphi(x-ct) + \varphi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\, d\xi$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\, d\xi$$

We will make use of the following facts to evaluate the integral.

Characteristic (or indicator) function of A

0. If $A \subseteq \mathbb{R}$ then $\chi_A(\xi) = \begin{cases} 1 & \text{if } \xi \in A, \\ 0 & \text{o.w.} \end{cases}$

1. $$\int_c^d f(\xi)\,d\xi = \int_{-\infty}^{\infty} \chi_{(c,d)}(\xi) f(\xi)\,d\xi$$

2. $\chi_A(\xi)\chi_B(\xi) = \chi_{A\cap B}(\xi)$ for all $A, B \subseteq \mathbb{R}$ and all $\xi \in \mathbb{R}$.

3. $$\int_{-\infty}^{\infty} \chi_{(c,d)}(\xi)\,d\xi = \text{length of } (c,d) \quad (= d-c).$$

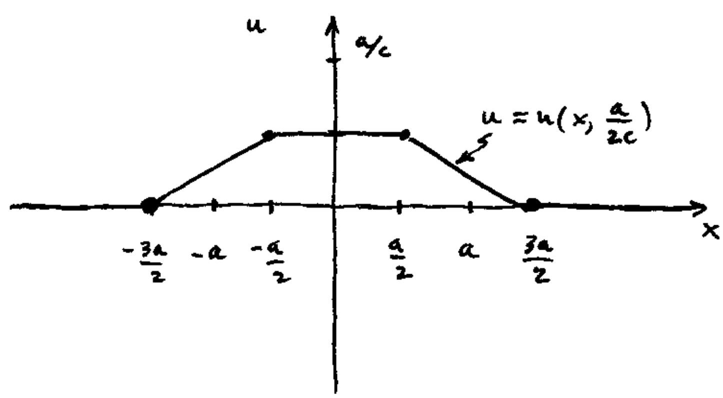
$$\therefore \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\,d\xi \overset{0.}{=} \frac{1}{2c}\int_{x-ct}^{x+ct} \chi_{(-a,a)}(\xi)\,d\xi$$

$$\overset{1.}{=} \frac{1}{2c}\int_{-\infty}^{\infty} \chi_{(x-ct,\,x+ct)}(\xi)\,\chi_{(-a,a)}(\xi)\,d\xi$$

$$\overset{2.}{=} \frac{1}{2c}\int_{-\infty}^{\infty} \chi_{(x-ct,\,x+ct)\cap(-a,a)}(\xi)\,d\xi$$

$$\overset{3.}{=} \frac{1}{2c}\,\text{length of}\,\{(x-ct,\,x+ct)\cap(-a,a)\}.$$

$\boxed{t = \frac{a}{2c}}$ $$u\left(x, \frac{a}{2c}\right) = \frac{1}{2c}\,\text{length of}\left\{\left(x-\frac{a}{2},\, x+\frac{a}{2}\right)\cap(-a,a)\right\}$$

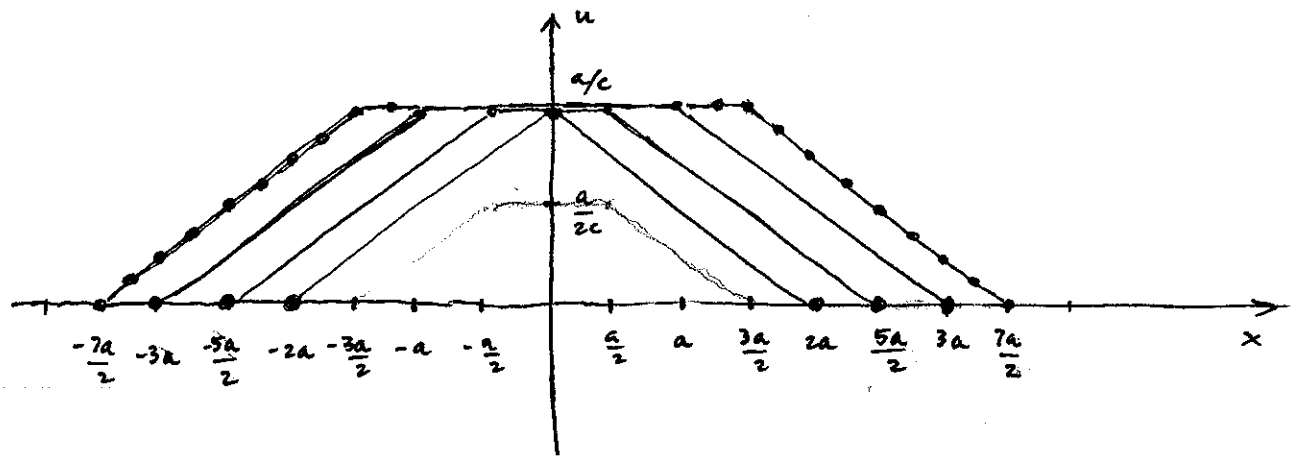


$\boxed{t = \frac{a}{c}}$ $\quad u(x, \frac{a}{c}) = \frac{1}{2c}$ length of $\{(x-a, x+a) \cap (-a,a)\}$

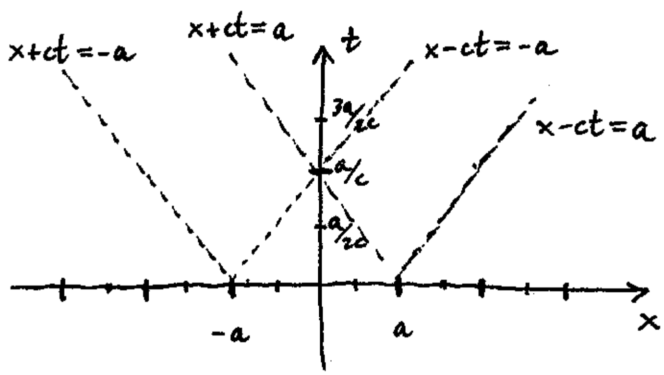
$\boxed{t = \frac{3a}{2c}}$ $\quad u(x, \frac{3a}{2c}) = \frac{1}{2c}$ length of $\{(x - \frac{3a}{2}, x + \frac{3a}{2}) \cap (-a,a)\}$

$\boxed{t = \frac{2a}{c}}$ $\quad u(x, \frac{2a}{c}) = \frac{1}{2c}$ length of $\{(x-2a, x+2a) \cap (-a,a)\}$

$\boxed{t = \frac{5a}{2c}}$ $\quad u(x, \frac{5a}{2c}) = \frac{1}{2c}$ length of $\{(x - \frac{5a}{2}, x + \frac{5a}{2}) \cap (-a,a)\}$



Notes: ① The function $u(x,t) = \frac{1}{2c}$ length of $\{(x-ct, x+ct) \cap (-a,a)\}$ is called a "generalized solution" of the problem: $u_{tt} - c^2 u_{xx} = 0$ in $-\infty < x < \infty, 0 < t < \infty$, and $u(x,0) = 0$ and $u_t(x,0) = \chi_{(-a,a)}(x)$ for $-\infty < x < \infty$. Note that $u = u(x,t)$ fails to be differentiable at all points on the lines $x - ct = \pm a$ and $x + ct = \pm a$, hence it is not a "classical solution"



of the problem (i.e. a function of x and t that:

(a) is C^2 and satisfies the PDE at all points of the region $-\infty < x < \infty$, $0 < t < \infty$;

(b) is C^1 on the closure of the region $-\infty < x < \infty$, $0 \le t < \infty$;

and (c) satisfies the initial conditions on the boundary $t = 0$ of the region.)

② The singularities of the generalized solution (i.e. points where the function fails to be differentiable) propagate along those characteristics $|x \pm ct| = a$ of the PDE $u_{tt} - c^2 u_{xx} = 0$ which pass through the points $(\pm a, 0)$ where the initial data is singular.