Sec. 2.2 Causality and Energy Conservation

The I.V.P.

$$(*)\quad \begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for } -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x) & \text{for } -\infty < x < \infty \end{cases}$$

has solution

$$u(x,t) = \frac{1}{2}\Big[\varphi(x-ct) + \varphi(x+ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds. \qquad \text{(d'Alembert's formula)}$$

Note that the value of the solution u of $(*)$ at (x_0,t_0) depends upon

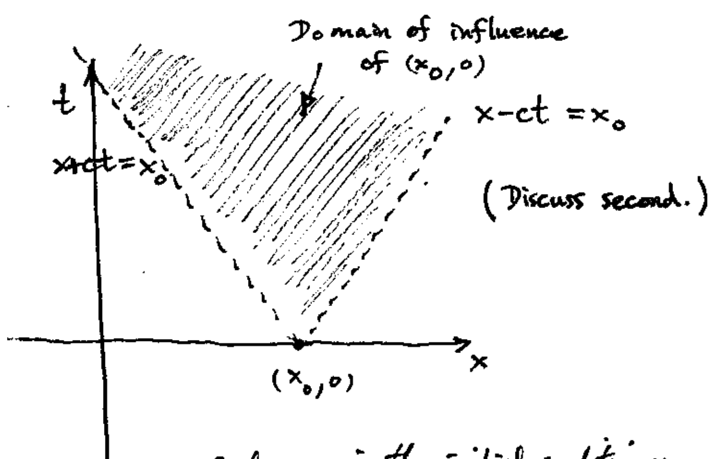
Information travels with speed c. → (1) the value of φ at the two points $x_0 \pm ct_0$, and

Information travels with speed $\le$ c. → (2) the values of ψ in the interval $[x_0 - ct_0,\ x_0 + ct_0]$.

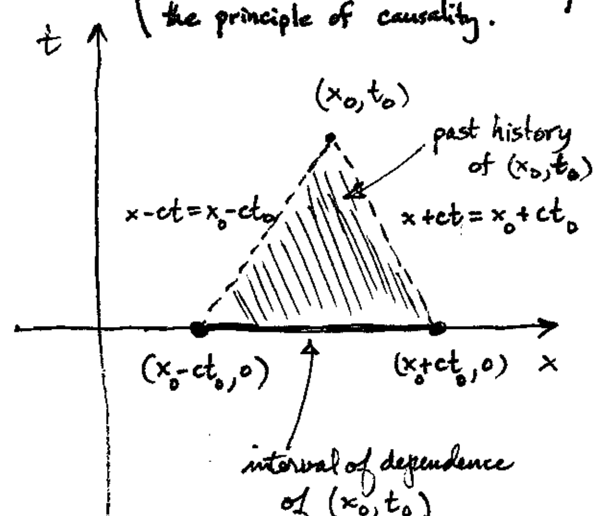
<u>Principle of Causality</u>: The value of the solution to the IVP$(*)$ at the point (x_0,t_0) depends only on the values $\varphi(x)$ and $\psi(x)$ where x belongs to the interval $|x - x_0| \le ct_0$.

[See Chap. 9: Sec. 1 for the 3-D version of causality
Sec. 3 for the relativistic version.] ← We see "sharp" images in 3-D; cf. Huygen's principle pp. 222–223.

(Discuss this first; it "illustrates" the principle of causality.)



A change in the initial conditions at $(x_0,0)$ can affect the solution of $(*)$ only in the domain of influence



Conservation of Energy in Waves

Let $u = u(x,t)$ be a classical solution to $\rho u_{tt} - T u_{xx} = 0$ in $-\infty < x < \infty$, $0 \le t < \infty$, satisfying the following decay conditions for each fixed $t \ge 0$:

(a) u_t, u_x, u_{tt}, u_{xx}, and u_{xt} are square-integrable functions of x on $(-\infty, \infty)$;

(b) $\lim_{|x| \to \infty} u_x(x,t)\, u_t(x,t) = 0$.

Then the total energy,

$$(*) \qquad E(t) = \left(\int_{-\infty}^{\infty} \left[\tfrac{1}{2} \rho u_t^2(x,t) + \tfrac{1}{2} T u_x^2(x,t) \right] dx \right)^{1/2},$$

of u at $t \ge 0$ is a constant, independent of t.

Notes: ① We will make use of the Cauchy-Schwarz inequality:

$$\int_a^b |f(x) g(x)|\, dx \le \left(\int_a^b |f(x)|^2 dx \right)^{1/2} \cdot \left(\int_a^b |g(x)|^2 dx \right)^{1/2}$$

which is valid for square-integrable functions f and g on (a,b). For a proof of this inequality see Rudin's "Principles of Mathematical Analysis" (third edition), Theorem 11.35.

② The basic idea of the proof is to show that the total energy function has a derivative that is identically zero. We will use a result on differentiating improper integrals: Theorem 2, Appendix A.3, p. 390 (first edition) or p. 420 (second edition).

__Proof of the Conservation of Energy Theorem for Waves__: Observe that for each fixed $t \geq 0$, the integrals $\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$ and $\int_{-\infty}^{\infty} u_x^2(x,t)\,dx$ exist and are finite by hypothesis. Therefore, the energy function E given by $(*)$ is defined and finite for each $t \geq 0$.

Apply the Cauchy-Schwarz inequality with $f(x) = u_t(x,t)$ and $g(x) = u_{tt}(x,t)$ to get that

$$\int_{-\infty}^{\infty} \left|\rho u_t(x,t) u_{tt}(x,t)\right| dx \leq \rho \left(\int_{-\infty}^{\infty} u_t^2(x,t)\,dx\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} u_{tt}^2(x,t)\,dx\right)^{1/2} < \infty,$$

and similarly

$$\int_{-\infty}^{\infty} \left|T u_x(x,t) u_{tx}(x,t)\right| dx \leq T \left(\int_{-\infty}^{\infty} u_x^2(x,t)\,dx\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} u_{tx}^2(x,t)\,dx\right) < \infty.$$

Therefore the hypotheses of Theorem 2 in Appendix A.3 are satisfied. Applying this theorem, we have

$$\frac{dE}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \left[\tfrac{1}{2}\rho u_t^2(x,t) + \tfrac{1}{2} T u_x^2(x,t)\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho u_t^2(x,t) + \tfrac{1}{2} T u_x^2(x,t)\right] dx$$

$$= \int_{-\infty}^{\infty} \left[\rho u_t(x,t) u_{tt}(x,t) + T u_x(x,t) u_{tx}(x,t)\right] dx. \qquad (**)$$

However, an integration by parts and application of hypothesis (b) yields

$$\int_{-\infty}^{\infty} T u_x(x,t) u_{tx}(x,t)\,dx = \lim_{\substack{M \to \infty \\ N \to -\infty}} \int_N^M \overbrace{T u_x(x,t)}^{U}\, \overbrace{u_{tx}(x,t)\,dx}^{dV}$$

$$= \lim_{\substack{M \to \infty \\ N \to -\infty}} \left(\cancelto{0}{T u_x(x,t) u_t(x,t) \Big|_{x=N}^{M}} - \int_N^M T u_{xx}(x,t) u_t(x,t)\,dx \right)$$

$$= - \int_{-\infty}^{\infty} T u_{xx}(x,t) u_t(x,t)\,dx .$$

Substituting this expression into (**) leads to

$$\frac{dE^2}{dt} = \int_{-\infty}^{\infty} \left[\rho u_t(x,t) u_{tt}(x,t) - T u_{xx}(x,t) u_t(x,t) \right] dx$$

$$= \int_{-\infty}^{\infty} u_t(x,t) \overbrace{\left[\rho u_{tt}(x,t) - T u_{xx}(x,t) \right]}^{0 \text{ since } u \text{ solves the wave eqn.}} dx$$

$$= 0 .$$

It follows from this that total energy of u is a constant, independent of t.

<u>Ex</u> (#1, p. 40) Use conservation of energy for solutions to the wave equation to show that the only C^2 solution to $\rho u_{tt} - T u_{xx} = 0$ for $-\infty < x < \infty$, $0 < t < \infty$, satisfying $u(x,0) = 0$, $u_t(x,0) = 0$ for $-\infty < x < \infty$, and the decay conditions (a) and (b) of the conservation of energy result, is $u(x,t) \equiv 0$.

Note: The result above is often referred to as a "uniqueness theorem". It implies that there can be at most one solution to the IVP

$$(***) \quad \begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & \text{for } -\infty < x < \infty,\ 0 < t < \infty, \\ u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x) & \text{for } -\infty < x < \infty. \end{cases}$$

(To see this, apply #1, p. 40 to the difference $u = u_1(x,t) - u_2(x,t)$ of two solutions to (***).)

Solution: Let $u=u(x,t)$ be a C^2 solution to the I.V.P. in #1, p.40. For all $t>0$ we have

$$(\dagger) \quad E(t) = E(0) = \int_{-\infty}^{\infty} \left[\tfrac{1}{2}\rho u_t^2(x,0) + \tfrac{1}{2} T u_x^2(x,0) \right] dx .$$

But $u_t(x,0) = 0$ for all $-\infty < x < \infty$, and $u(x,0) = 0$ for all $-\infty<x<\infty$ implies

$$u_x(x,0) = \lim_{h\to 0} \frac{u(x+h,0) - u(x,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

for all $-\infty<x<\infty$. Hence $(\dagger)$ implies

$$0 = E(t) = \int_{-\infty}^{\infty} \left[\tfrac{1}{2}\rho u_t^2(x,t) + \tfrac{1}{2} T u_x^2(x,t) \right] dx$$

for all $t>0$. But the integrand of $E(t)$ is clearly nonnegative and continuous for all $-\infty<x<\infty$ and $0<t<\infty$, so the Vanishing Theorem (Appendix A.1, p.385) implies $\tfrac{1}{2}\rho u_t^2(x,t) + \tfrac{1}{2} T u_x^2(x,t) = 0$ for $-\infty<x<\infty$ and each $t>0$.

$\therefore \quad u_t(x,t) = u_x(x,t) = 0$ for all $-\infty < x < \infty$ and each $t>0$.

It follows that $u(x,t) = \text{constant}$ for $-\infty<x<\infty$ and $0<t<\infty$.

But $u(x,0) \equiv 0$ so $u(x,t) \equiv 0$.