Secs 2.1–2.2 were devoted to solving the I.V.P.

$$(*)\quad \begin{cases} u_{tt} - c^2 u_{xx} \overset{①}{=} 0 & \text{for } -\infty<x<\infty,\ 0<t<\infty, \\ u(x,0) \overset{②}{=} \varphi(x) \text{ and } u_t(x,0) \overset{③}{=} \psi(x) & \text{for } -\infty<x<\infty. \end{cases}$$

arbitrary

Step 1: Find the general solution to ① : $u(x,t) = f(x-ct) + g(x+ct)$.

Step 2: Determine f and g so that ② and ③ are satisfied.

Secs. 2.3–2.4 will be devoted to solving the I.V.P.

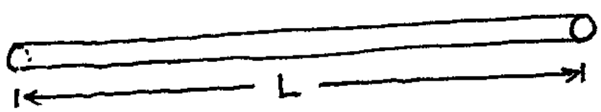
$$(**)\quad \begin{cases} u_t - k u_{xx} \overset{①}{=} 0 & \text{for } -\infty<x<\infty,\ 0<t<\infty, \\ u(x,0) \overset{②}{=} \varphi(x) & \text{for } -\infty<x<\infty. \end{cases}$$

~~**Step 1**: Find the general solution to ①.~~ ← Apparently this is an unsolved problem.

Instead, Strauss will take an indirect approach to solving (**). In Sec. 2.3, general properties of solutions to $u_t - k u_{xx} = 0$ will be developed. These will be applied in Sec. 2.4 to solve (**).

Sec. 2.3 The Diffusion Equation

Consider a thin rod of length L whose lateral sides are insulated,

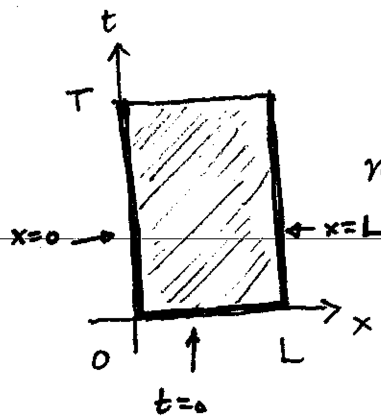


and whose initial temperature distribution and temperatures at the ends are known. The temperature u at position x and time t in the rod obeys

$$\begin{cases} u_t - k u_{xx} = 0 & \text{for } 0 < x < L,\ 0 < t \le T, \\ u(x,0) = \varphi(x) & \text{for } 0 \le x \le L, \\ u(0,t) = g(t) \text{ and } u(L,t) = h(t) & \text{for } 0 \le t \le T. \end{cases}$$

Q: When and where do the hottest and coldest temperatures in the rod occur?

(Physical Reasoning) A. Either (1) initially (i.e. $t=0$), or
(2) at the ends of the rod (i.e. $x=0$ or $x=L$).



This is the content of the weak maximum/minimum principle (see handout on next page). We will establish this principle through purely mathematical arguments, without recourse to physical reasoning.

<u>Basic Idea of the proof</u>: Suppose that u attained a maximum at an interior point (x_0,t_0) of R. Then $u_t(x_0,t_0)=0$ and $u_{xx}(x_0,t_0) \le 0$. If this inequality were strict then

$$0 = u_t(x_0,t) - k u_{xx}(x_0,t_0) > 0,$$

clearly a contradiction. However $u_{xx}(x_0,t_0) = 0$ is possible, so we must argue more carefully.

To here Day 11. No lecture on Day 12 --- Exam I.

Mathematics 325
Partial Differential Equations
Sec. 2.3: The Diffusion Equation

<u>(Weak) Maximum Principle:</u> Let $u = u(x,t)$ be a solution to the diffusion equation

$$u_t - ku_{xx} = 0$$

in the rectangle R: $0 < x < L$, $0 < t \le T$, and let u be continuous on the closed rectangle $\overline{R} = R \cup \partial R$. Then the maximum value of u on $\overline{R}$ is attained either initially (i.e. when $t = 0$) or on the lateral walls (i.e. where $x = 0$ or $x = L$).

Notes:

1. The above principle holds with "maximum" replaced by "minimum".
2. A stronger version of the maximum principle holds (see p. 41 of Strauss), but the weak version above is usually sufficient for applications.

<u>Uniqueness Theorem:</u> There is at most one solution $u = u(x,t)$ to

$$u_t - ku_{xx} = f(x,t)$$

in the rectangle R: $0 < x < L$, $0 < t \le T$, satisfying the initial and boundary conditions

$$u(x,0) = \phi(x) \quad \text{for } 0 \le x \le L,$$
$$u(0,t) = g(t) \text{ and } u(L,t) = h(t) \quad \text{for } 0 \le t \le T,$$

and such that u is continuous on $\overline{R}$.

<u>Stability Theorem:</u> Let $u = u_1(x,t)$ and $u = u_2(x,t)$ be solutions to

$$u_t - ku_{xx} = f(x,t)$$

in R: $0 < x < L$, $0 < t < \infty$ such that

(1) $u_j(x,0) = \phi_j(x)$ for $0 \le x \le L$ $(j = 1,2)$,

(2) $u_j(0,t) = g(t)$ and $u_j(L,t) = h(t)$ for $t \ge 0$ $(j = 1,2)$, and

(3) u_j is continuous on $\overline{R}$ $(j = 1,2)$.

Then, for any $t > 0$,

$$(4) \quad \int_0^L |u_1(x,t) - u_2(x,t)|^2 dx \le \int_0^L |\phi_1(x) - \phi_2(x)|^2 dx$$

and

$$(5) \quad \max_{0 \le x \le L} |u_1(x,t) - u_2(x,t)| \le \max_{0 \le x \le L} |\phi_1(x) - \phi_2(x)| .$$

Let u be a solution to the diffusion equation in R and let u be continuous on $\overline{R} = R \cup \partial R$.

Alternate *Proof of the Weak Maximum Principle*: Let $\varepsilon > 0$ and define

$$v(x,t) = u(x,t) + \varepsilon x^2. \quad \left(\begin{array}{l}\text{A perturbation term designed} \\ \text{so } v_{xx}(x_0,t_0) < 0.\end{array}\right)$$

Since v is continuous on the closed, bounded rectangle $\overline{R}$, v attains a maximum value at some point (x_0,t_0) in $\overline{R}$.

Suppose (x_0,t_0) belongs to R.

Can't happen → Case 1: Suppose $0 < x_0 < L$ and $0 < t_0 < T$. (I.e. (x_0,t_0) is an interior point of $\overline{R}$.) Then $v_t(x_0,t_0) = 0$ and $v_{xx}(x_0,t_0) \le 0$. This implies $u_t(x_0,t_0) = 0$ and $u_{xx}(x_0,t_0) + 2\varepsilon \le 0 \Rightarrow u_{xx}(x_0,t_0) \le -2\varepsilon < 0$. Then

$$0 = u_t(x_0,t_0) - k u_{xx}(x_0,t_0) \ge 0 + 2k\varepsilon > 0, \text{ a contradiction.}$$

Can't happen → Case 2: Suppose $0 < x_0 < L$ and $t_0 = T$. (I.e. (x_0,t_0) is on the "back wall" $t = T$.) Then $v_t(x_0,T) \ge 0$ and $v_{xx}(x_0,T) \le 0$. This implies $u_t(x_0,T) \ge 0$ and $u_{xx}(x_0,T) + 2\varepsilon \le 0 \Rightarrow u_{xx}(x_0,T) \le -2\varepsilon < 0$. Again we have a contradiction:

$$0 = u_t(x_0,T) - k u_{xx}(x_0,T) \ge 0 + 2k\varepsilon > 0.$$

This shows that $(x_0,t_0) \in \overline{R} \setminus R$, i.e. $\max\limits_{(x,t)\in\overline{R}} v(x,t) = \max\limits_{(x,t)\in\overline{R}\setminus R} v(x,t)$.

Because $v(x,t) = u(x,t) + \varepsilon x^2 \ge u(x,t)$,

$$\max_{(x,t)\in\overline{R}} u(x,t) \le \max_{(x,t)\in\overline{R}} v(x,t) \overset{\text{(See Argument Above)}}{=} \max_{(x,t)\in\overline{R}\setminus R} v(x,t) = \max_{(x,t)\in\overline{R}\setminus R} \left\{ u(x,t) + \varepsilon x^2 \right\}$$

$$\le \max_{(x,t)\in\overline{R}\setminus R} u(x,t) + \max_{(x,t)\in\overline{R}\setminus R} \varepsilon x^2 = \varepsilon L^2 + \max_{(x,t)\in\overline{R}\setminus R} u(x,t)$$

But $\varepsilon > 0$ is arbitrary, so letting $\varepsilon \to 0^+$ we have

$$\max_{(x,t)\in\overline{R}} u(x,t) \le \max_{(x,t)\in\overline{R}\setminus R} u(x,t).$$

The reverse inequality $\max_{(x,t)\in\overline{R}\setminus R} u(x,t) \le \max_{(x,t)\in\overline{R}} u(x,t)$ is clearly true. Thus $\max_{(x,t)\in\overline{R}} u(x,t) = \max_{(x,t)\in\overline{R}\setminus R} u(x,t)$. QED.

To here Day 13. Spent ½ hour working problems from Exam I, etc.

omit if pressed for time

Re Note 1: Let u be any continuous function on $\overline{R}$ satisfying $u_t - ku_{xx} = 0$ in R. Then the same is true of the function $-u$ so, by the weak maximum principle, $\max_{(x,t)\in\overline{R}} \{-u(x,t)\} = \max_{(x,t)\in\overline{R}\setminus R} \{-u(x,t)\}$. However

$$\min_{z\in S} f(z) = -\max_{z\in S}\{-f(z)\}$$

is true for any bounded, real-valued function f on any set S. Therefore

$$\min_{(x,t)\in\overline{R}} u(x,t) = -\max_{(x,t)\in\overline{R}}\{-u(x,t)\} = -\max_{(x,t)\in\overline{R}\setminus R}\{-u(x,t)\} = \min_{(x,t)\in\overline{R}\setminus R} u(x,t)$$

I.e. the principle holds with "minimum" in place of "maximum".

The uniqueness theorem follows easily from the weak maximum/minimum principle. For suppose u_1 and

u_2 are continuous functions on $\overline{R}$ which solve

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{in } R:\ 0<x<L,\ 0<t\le T, \\ u(x,0) = \varphi(x) & \text{for } 0 \le x \le L, \\ u(0,t) = g(t) \text{ and } u(L,t) = h(t) & \text{for } 0 \le t \le T. \end{cases}$$

Then $u(x,t) = u_1(x,t) - u_2(x,t)$ is a continuous function on $\overline{R}$ which solves $u_t - ku_{xx} = 0$ in R and has the property that $u(x,t) \equiv 0$ if $(x,t) \in \overline{R} \setminus R$. Hence

$$0 = \min_{(x,t)\in\overline{R}\setminus R} u(x,t) = \min_{(x,t)\in\overline{R}} u(x,t) \le \max_{(x,t)\in\overline{R}} u(x,t) = \max_{(x,t)\in\overline{R}\setminus R} u(x,t) = 0$$

Thus $u_1(x,t) - u_2(x,t) = u(x,t) \equiv 0$ on $\overline{R}$, i.e. $u_1 = u_2$ on $\overline{R}$.

<u>Ex</u> (#4, pp. 44-45) Let $u = u(x,t)$ be the solution to

$$(*) \begin{cases} u_t - u_{xx} = 0 & \text{for } 0<x<1,\ 0<t<\infty, \\ u(0,t) = 0 \text{ and } u(1,t) = 0 & \text{for } 0 \le t < \infty, \\ u(x,0) = 4x(1-x) & \text{for } 0 \le x \le 1 \end{cases}$$

(a) Show that $0 < u(x,t) < 1$ for all $0 < x < 1$ and all $t > 0$.

(b) Show that $u(x,t) = u(1-x,t)$ for all $0 \le x \le 1$ and all $t \ge 0$.

(c) Show that $\Theta(t) = \underbrace{\left(\int_0^1 u^2(x,t)\,dx \right)^{1/2}}_{\text{root mean square temperature of the rod at time } t}$ is a strictly decreasing function of t.

Discuss this result from a physical point of view. Why do we expect the root mean square temperature of the rod to strictly decrease with time?

Solution:

$$\max_{0\le t\le T} u(0,t) = 0 = \max_{0\le t\le T} u(L,t), \quad \max_{0\le x\le 1} u(x,0) = \max_{0\le x\le 1} 4x(1-x) = 1$$

$$\min_{0\le t\le T} u(0,t) = 0 = \min_{0\le t\le T} u(L,t), \quad \min_{0\le x\le 1} u(x,0) = \min_{0\le x\le 1} 4x(1-x) = 0$$

By the weak maximum/minimum principle, $0 \le u(x,t) \le 1$ for all $0\le x\le 1$ and $0\le t\le T$. In order to show the strict inequality $0<u(x,t)<1$ for all $0<x<1$ and $0<t\le T$, we must invoke the strong maximum/minimum principle (cf. p.41 middle). Note that $u=u(x,t)$ cannot be a constant function in the rectangle $0\le x\le 1, 0\le t\le T$, because $u(x,0)=4x(1-x)$ for $0\le x\le 1$.

(b) Let $v(x,t)=u(1-x,t)$ at each point (x,t) in $0<x<1$, $0<t<\infty$. It is routine to check that v is a solution to (*). The uniqueness theorem implies that $v(x,t)=u(x,t)$ for all $0\le x\le 1$ and $0\le t<\infty$,

i.e. $u(1-x,t)=u(x,t)$ " " " " " .

(c) $$\frac{d\Theta^2}{dt} = \frac{d}{dt}\int_0^1 u^2(x,t)\,dx = \int_0^1 \frac{\partial}{\partial t}\{u^2(x,t)\}\,dx = \int_0^1 2u(x,t)u_t(x,t)\,dx$$

$$2\Theta(t)\frac{d\Theta}{dt}$$

$$= \int_0^1 \overbrace{2u(x,t)}^{v}\overbrace{u_{xx}(x,t)\,dx}^{dV} = \cancel{2u(x,t)u_x(x,t)\Big|_{x=0}^{x=1}} - 2\int_0^1 u_x^2(x,t)\,dx$$

We claim that $x\mapsto u_x(x,t)$ is a nonconstant $\underbrace{\text{continuous}}_{\text{clear}}$ function of x for each fixed $t>0$. To see this, suppose that $x\longmapsto u_x(x,t_0)$

is constant as a function of x for some $t_0 > 0$. Then

$$(\dagger) \qquad u(x,t_0) - \underbrace{u(0,t_0)}_{0} = \int_0^x \overbrace{u_x(\xi,t_0)}^{c}\, d\xi = cx \qquad \text{for } 0 \le x \le 1.$$

Take $x = 1$ in $(\dagger)$ and obtain $c = \overbrace{u(1,t_0)}^{0} - \overbrace{u(0,t_0)}^{0} = 0$;

that is, $u(x,t_0) = 0$ for all $0 < x < 1$, contradicting part (a).

We conclude that

$$2\Theta(t) \cdot \frac{d\Theta}{dt} = -2 \int_0^1 \underbrace{u_x^2(x,t)}_{\substack{\text{not identically zero,} \\ \text{continuous}}}\, dx \quad < \quad 0 \ .$$

~~~~ To here Day 14 ~~~~

<u>Remarks on the Stability Theorem</u>: If f and g are continuous functions on $[a,b]$, define

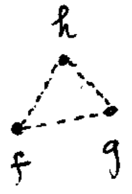
$$d_\infty(f,g) = \max_{a \le x \le b} |f(x) - g(x)|$$

$$d_2(f,g) = \left(\int_a^b \big(f(x) - g(x)\big)^2 dx \right)^{1/2}.$$

These define <u>metrics</u> on the space $C[a,b]$ of all continuous real-valued functions on $[a,b]$; i.e. they both satisfy

(1) $d(f,g) \ge 0$ for all $f, g \in C[a,b]$, with equality only if $f = g$;

(2) $d(f,g) = d(g,f)$ for all $f, g \in C[a,b]$;



(3) $d(f,g) \le d(f,h) + d(h,g)$ for all $f, g, h \in C[a,b]$.

triangle inequality
~~~~

With this notation, the conclusion

$$(4) \qquad \left(\int_0^L |u_1(x,t) - u_2(x,t)|^2 dx \right)^{1/2} \le \left(\int_0^L |\varphi_1(x) - \varphi(x_2)|^2 dx \right)^{1/2} \quad (t>0)$$

of the stability theorem can be rewritten as

$$(4) \qquad d_2(u_1(\cdot,t), u_2(\cdot,t)) \le d_2(\varphi_1, \varphi_2) \qquad (t>0)$$

That is, for each fixed $t>0$, the distance between u_1 and u_2 in the d_2-metric is no greater than the distance between the initial data φ_1 and φ_2. (A similar statement holds for the d_∞-metric in (5).) Thus, the solution to $u_t - ku_{xx} = f(x,t)$ in R satisfying (1)-(2) varies continuously with the initial data; i.e. such (forward) diffusion problems are "well-posed".