Sec. 2.4 Diffusion in the Upper Half Plane

Main Result: Let $\varphi = \varphi(x)$ be a bounded continuous function on $(-\infty, \infty)$.

Then

$$(\dagger) \quad u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} \varphi(y)\,dy \qquad (\text{for } t>0 \text{ and } -\infty<x<\infty)$$

is a solution to the I.V.P.

$$(*) \quad \begin{cases} u_t - k u_{xx} = 0 & \text{for } -\infty<x<\infty,\ 0<t<\infty \\ u(x,0) = \varphi(x) & \text{for } -\infty<x<\infty \end{cases}$$

in the ~~following~~ sense:

$$\lim_{t \to 0^+} u(x,t) = \varphi(x) \qquad \text{for } -\infty<x<\infty.$$

Notes:

① If φ is merely piecewise continuous and bounded on $(-\infty, \infty)$ then $(\dagger)$ defines a solution to $(*)$ in the ~~following~~ sense that

$$\lim_{t \to 0^+} u(x,t) = \frac{\varphi(x^+) + \varphi(x^-)}{2} \qquad \text{for } -\infty<x<\infty.$$

(For a rigorous proof of this, see Sec. 3.5, pp. 79-80 and exercises #1, 2 on p. 81.)

② Strauss uses techniques from the theory of distributions to "derive" formula (†) from (*). (See pp. 45–49) We will use Fourier transform methods to "derive" (†) from (*).

③ The claim of uniqueness on pp. 47–48 of Strauss for $u = u(x,t)$ in (†) as the solution to (*) ~~on pp. 47–48 of Strauss~~ is FALSE. See Fritz John, <u>Partial Differential Equations</u>, 4th ed., Springer-Verlag, New York, 1982, pp. 211–213.

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Properties of the Gaussian Heat Kernel :

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}}\, e^{-\frac{x^2}{4kt}} \qquad (t>0,\ -\infty<x<\infty).$$

Routine ⓪ $u = S(x,t)$ is a solution to $u_t - k u_{xx} = 0$ for $t>0$ and $-\infty<x<\infty$.

Clear ① $S(x,t) > 0$ for all $-\infty<x<\infty$ and $t>0$.

Lecture ② $\displaystyle\int_{-\infty}^{\infty} S(x,t)\,dx = 1$ for all $t>0$.

(Sec. 2.4, #8) ③ For any $\delta>0$, $\displaystyle\max_{\delta<|x|<\infty} S(x,t) \longrightarrow 0$ as $t\to 0^+$.

Properties ①, ②, and ③ ensure that the family of functions $\{S(\cdot,t)\}_{t>0}$ is an approximate identity for the convolution algebra $L^1(\mathbb{R})$:

$$\lim_{t\to 0^+} \| f - f * S_t \|_{L^1(\mathbb{R})} = 0 \qquad \text{for all } f\in L^1(\mathbb{R}).$$
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Notation:

$$L^1(\mathbb{R}) = \left\{ \underset{\text{measurable}}{f : \mathbb{R} \to \mathbb{R}} \;\middle|\; \int_{-\infty}^{\infty} |f(x)|\,dx < \infty \right\}$$

$$\|f\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |f(x)|\,dx < \infty$$

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \qquad \left(f,g \in L^1(\mathbb{R}),\ x \in \mathbb{R} \right)$$

Property ⓪ ensures that

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4kt}}}{\sqrt{4k\pi t}} \varphi(y)\,dy = \left(S(\cdot,t) * \varphi \right)(x)$$

is a solution to $u_t - ku_{xx} = 0$ in the upper halfplane: $-\infty < x < \infty$, $0 < t < \infty$.

Ex 1 (p.51, #6) Show that $\displaystyle\int_{-\infty}^{\infty} S(x,t)\,dx = \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{4kt}}}{\sqrt{4k\pi t}}\,dx = 1$ for all $t > 0$.
(Property ②)

We first show $\displaystyle I = \int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}$. To see this, observe that

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-p^2}\,dp \right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx \right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy \right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy.$$

We use polar coordinates: $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan\left(\frac{y}{x}\right) \end{cases}$. Recall that $dx\,dy = r\,dr\,d\theta$.

$$\therefore\ I^2 = \int_0^{2\pi} -\frac{1}{2}\int_0^{\infty} e^{-r^2}\,{-2r}\,dr\,d\theta = \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_{r=0}^{\infty}\,d\theta = \int_0^{2\pi} \frac{1}{2}\,d\theta = \pi. \Rightarrow I = \sqrt{\pi}$$

$$\therefore \int_{-\infty}^{\infty} S(x,t)\,dx = \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{4kt}}}{\sqrt{4k\pi t}}\,dx$$

Let $p = \frac{x}{\sqrt{4kt}}$

Then $dp = \frac{dx}{\sqrt{4kt}}$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\,dp$$

$$= 1$$

<u>Ex 2</u> (#3, p.51) Solve $u_t - ku_{xx} = 0$ in $-\infty < x < \infty$, $0 < t < \infty$, subject to $u(x,0) = e^{3x}$ for $-\infty < x < \infty$.

A candidate for a solution is

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} e^{3y}\,dy \qquad (\text{for } t>0 \text{ and } -\infty<x<\infty).$$

(Note that $\varphi(x) = e^{3x}$ is continuous but not bounded on $(-\infty,\infty)$, so we are "abusing" formula (†). We will need to check our answer when we are finished calculating the solution.)

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt} + 3y}\,dy$$

We complete the square in y in the exponent:

$$-\frac{(x-y)^2}{4kt} + 3y = -\frac{\left[x^2 - 2xy + y^2 - 12kty\right]}{4kt}$$

$$= -\frac{\left[y^2 - 2(6kt+x)y + (6kt+x)^2\right]}{4kt} + \frac{(6kt+x)^2 - x^2}{4kt}$$

$$= \frac{-(y-6kt-x)^2}{4kt} + \frac{36k^2t^2 + 12ktx + \cancel{x^2} - \cancel{x^2}}{4kt}$$

$$= \frac{-(y-6kt-x)^2}{4kt} + 9kt + 3x$$

$$\therefore \quad u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(y-6kt-x)^2}{4kt}} \cdot e^{3x+9kt}\, dy$$

Let $p = \frac{y-6kt-x}{\sqrt{4kt}}$

Then $dp = \frac{dy}{\sqrt{4kt}}$

$$= \frac{e^{3x+9kt}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\, dp$$

$$= \boxed{e^{3x+9kt}}$$

Check: $u_t - ku_{xx} = 9ke^{3x+9kt} - k\left(9e^{3x+9kt}\right) \overset{\checkmark}{=} 0$

$u(x,0) \overset{\checkmark}{=} e^{3x}$

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(#4, p.51)

Ex 3 Solve $u_t - ku_{xx} = 0$ for $-\infty < x < \infty$, $0 < t < \infty$, subject to

$$u(x,0) = \varphi(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

A solution is given by

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\, dy = \frac{1}{\sqrt{4k\pi t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot e^{-y}\, dy$$

$$-\frac{(x-y)^2}{4kt} - y = \frac{-\left[x^2-2xy+y^2+4kty\right]}{4kt}$$

$$= \frac{-\left[y^2+2(2kt-x)y+(2kt-x)^2\right]}{4kt} + \frac{(2kt-x)^2-x^2}{4kt}$$

$$= -\frac{(y+2kt-x)^2}{4kt} + \frac{4k^2t^2-4ktx+\cancel{x^2}-\cancel{x^2}}{4kt}$$

$$= -\frac{(y+2kt-x)^2}{4kt} + kt-x$$

Omit details in class!?

By a routine completion of the square on y in the exponent, we have

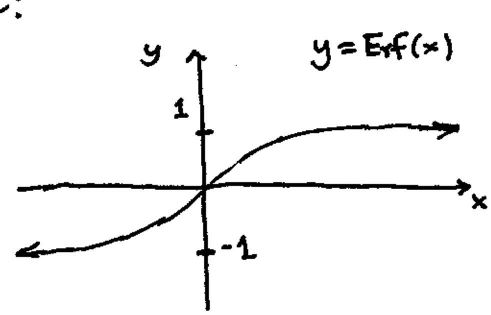
$$u(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_0^\infty e^{kt-x}\cdot e^{-\frac{(y+2kt-x)^2}{4kt}}\,dy.$$

Let $z = \dfrac{y+2kt-x}{\sqrt{4kt}}$. Then $dz = \dfrac{dy}{\sqrt{4kt}}$

$$= \frac{e^{kt-x}}{\sqrt{\pi}}\int_{\frac{2kt-x}{\sqrt{4kt}}}^{\infty} e^{-z^2}\,dz$$

Note: There is no elementary antiderivative for e^{-z^2}. The answer can be expressed in terms of the error function, however:

$$\operatorname{Erf}(w) = \frac{2}{\sqrt{\pi}}\int_0^w e^{-p^2}\,dp$$



Note that $\dfrac{\sqrt{\pi}}{2} = \displaystyle\int_0^\infty e^{-p^2}dp = \int_0^w e^{-p^2}dp + \int_w^\infty e^{-p^2}dp$ so

$$\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2}\,\mathrm{Erf}(w) + \int_w^\infty e^{-p^2}\,dp \quad \Rightarrow \quad \int_w^\infty e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}\left(1 - \mathrm{Erf}(w)\right).$$

$$\therefore\; u(x,t) = \frac{e^{kt-x}}{\cancel{\sqrt{\pi}}} \cdot \frac{\cancel{\sqrt{\pi}}}{2}\left(1 - \mathrm{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right)\right)$$

$$= \boxed{\frac{e^{kt-x}}{2}\left(1 - \mathrm{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right)\right)}$$

Challenge: Show

Check: $\displaystyle \lim_{t\to 0^+} u(x,t) = \frac{e^{-x}}{2}\lim_{t\to 0^+}\left[1 - \mathrm{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right)\right]$

$$= \frac{e^{-x}}{2}\cdot\begin{cases} 2 & \text{if } x>0, \\ 0 & \text{if } x<0, \end{cases}$$

$$= \begin{cases} e^{-x} & \text{if } x>0, \\ 0 & \text{if } x<0. \end{cases}$$

$\mu = 0,\ \sigma^2 = .5$ HP-49G (see next page)

$\mathrm{Erf}(w) = 1 - 2\,\mathrm{UTPN}(w)$

Sample calculation assuming $k=1$.

$$u(2,3) = \frac{e^{3-2}}{2}\left(1 - \mathrm{Erf}\left(\frac{6-2}{\sqrt{12}}\right)\right) \doteq \frac{e}{2}\Big(1 - \overbrace{\mathrm{Erf}(1.1547)}^{.897529}\Big) \doteq 0.139272$$

Q: How does one calculate values of Erf(w) using the HP-49G ?

$$\mathrm{Erf}(w) = \frac{2}{\sqrt{\pi}} \int_0^{w} e^{-p^2}\,dp$$

On the HP-49G, if:

(level 3) μ

(level 2) σ^2

(level 1) x

then pressing UTPN gives the probability that a normal random variable with mean μ and variance σ^2 is greater than x. That is,

$$\mathrm{UTPN}(x) = \int_x^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt.$$

If we take $\boxed{\mu = 0 \text{ and } \sigma^2 = 1/2}$ then

$$\mathrm{UTPN}(x) = \frac{1}{\sqrt{\pi}} \int_x^{\infty} e^{-p^2}\,dp .$$

From the identity $\int_x^{\infty} e^{-p^2}dp = \int_0^{\infty} e^{-p^2}dp - \int_0^{x} e^{-p^2}dp$ we have

$$\sqrt{\pi}\,\mathrm{UTPN}(x) = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\mathrm{Erf}(x) \Rightarrow \frac{1}{2}\mathrm{Erf}(x) = \frac{1}{2} - \mathrm{UTPN}(x)$$

$$\Rightarrow \boxed{\mathrm{Erf}(w) = 1 - 2*\mathrm{UTPN}(w)}.$$