Sec 2.5 Comparison of Waves and Diffusions

Summary:

Wave I.V.P. $\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for } -\infty < x < \infty,\ -\infty < t < \infty. \\ u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x) & \text{for } -\infty < x < \infty. \end{cases}$

(*) Solution: $$u(x,t) = \frac{1}{2}\left[\varphi(x-ct) + \varphi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

Diffusion I.V.P. $\begin{cases} u_t - k u_{xx} = 0 & \text{for } -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) = \varphi(x) & \text{for } -\infty < x < \infty \end{cases}$

(**) Solution: $$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4kt}}}{\sqrt{4k\pi t}}\,\varphi(y)\,dy$$

State if Sec. 2.5 is studied outside class.

You are responsible for knowing the properties of the solutions of the wave and diffusion I.V.P.'s given in the table on p. 53. (p.   in second edition)

Property	Waves	Diffusions
(i) speed of propagation	finite (and $\leq c$) (Apparent from d'Alembert's formula (*).)	infinite (Apparent from formula (**).)
(v) maximum principle	No (Consider $u(x,t) = \sin(\pi x)\sin(\pi t)$. u solves $u_{tt} - u_{xx} = 0$ in R: $0 \leq x \leq 1$, $0 \leq t \leq 1$. However $u = 0$ on ∂R while $u(\frac{1}{2}, \frac{1}{2}) = 1$.)	yes (Cf. p. 41, theorem)
(vi) behavior as $t \to \infty$?	$u(x,t) \not\to 0$ as $t \to \infty$	$u(x,t) \to 0$ as $t \to \infty$ (provided φ is absolutely integrable)

[over]

Suppose $u = u(x,t)$ solves the diffusion I.V.P. in the upper halfplane and suppose $\int_{-\infty}^{\infty} |\varphi(y)|\,dy < \infty$. Then

$$|u(x,t)| = \left| \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4kt}}}{\sqrt{4k\pi t}}\,\varphi(y)\,dy \right|$$

If $a \geq 0$ then $0 < e^{-a} \leq 1$.

$$\leq \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} |\varphi(y)|\,dy$$

$$\leq \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} |\varphi(y)|\,dy \longrightarrow 0 \quad \text{as} \quad t \to \infty.$$