Sec. 5.2 Even, Odd, Periodic, and Complex Functions

In the example for Sec. 4.2 (cf. #3, p. 90) we modeled diffusion inside a closed narrow circular tube of length $2l$:

$$\begin{cases} u_t - k u_{xx} \overset{①}{=} 0 & \text{for } -l<x<l,\ 0<t<\infty \\ u(-l,t) \overset{②}{=} u(l,t) \text{ and } u_x(-l,t) \overset{③}{=} u_x(l,t) & \text{for } t\geq 0, \\ u(x,0) \overset{④}{=} \cos^2\left(\frac{\pi x}{l}\right) & \text{for } -l \leq x \leq l. \end{cases}$$

When we separated variables, we found that the homogeneous portion ①-②-③ led to eigenvalues and eigenfunctions

$$\lambda_0 = 0 \quad , \qquad X_0(x) = 1$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad X_n(x) = \cos\left(\frac{n\pi x}{l}\right),\ \tilde{X}_n(x) = \sin\left(\frac{n\pi x}{l}\right) \quad (n=1,2,3,\dots)$$

The complete set of eigenfunctions

$$\Phi = \left\{1,\ \cos\left(\frac{\pi x}{l}\right),\ \sin\left(\frac{\pi x}{l}\right),\ \cos\left(\frac{2\pi x}{l}\right),\ \sin\left(\frac{2\pi x}{l}\right),\ \dots\right\} \text{ for this problem}$$

forms an orthogonal set on $(-l, l)$. The Fourier coefficients of a square integrable function f on $(-l, l)$ with respect to Φ are called the <u>full real Fourier coefficients of f</u> and the resulting Fourier series of f with respect to Φ is called the <u>full real Fourier series of f</u>. (Refer to the handout "Examples for Sec. 5.2" for the formal definitions and the statement of example 1. See next page.)

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Examples for Section 5.2

Definition. Let f be a $2\ell-$periodic function which is square-integrable on any period. The Fourier coefficients of f with respect to the orthogonal set

$$\left\{ 1,\ \cos\left(\frac{\pi x}{\ell}\right),\ \sin\left(\frac{\pi x}{\ell}\right),\ \cos\left(\frac{2\pi x}{\ell}\right),\ \sin\left(\frac{2\pi x}{\ell}\right),\ \ldots \right\}$$

on $(-\ell,\ell)$ are called the full real Fourier coefficients of f and are given by

$$a_0 = \frac{\langle f,1\rangle}{\langle 1,1\rangle} = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx,$$

and, for $n = 1,2,3,\ldots$, by

$$a_n = \frac{\langle f,\cos(n\pi(\bullet)/\ell)\rangle}{\langle \cos(n\pi(\bullet)/\ell),\cos(n\pi(\bullet)/\ell)\rangle} = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx,$$

$$b_n = \frac{\langle f,\sin(n\pi(\bullet)/\ell)\rangle}{\langle \sin(n\pi(\bullet)/\ell),\sin(n\pi(\bullet)/\ell)\rangle} = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx.$$

The full real Fourier series of f is

$$a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{\ell}\right) + b_n\sin\left(\frac{n\pi x}{\ell}\right)\right].$$

Example 1. Consider the 1-periodic function f determined by

$$f(x) = 2x^4 - x^2 \quad \text{if} \quad -1/2 \le x < 1/2.$$

(a) Find the full real Fourier series of f on the interval $(-1/2,1/2)$.

(b) Assuming that the Fourier series of f at x converges to $f(x)$ for all real x (cf. Theorem 4 of Section 5.4), show that

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

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Definition. Let f be a 2ℓ – periodic function which is square-integrable on any period. The Fourier coefficients of f with respect to the orthogonal set $\left\{e^{in\pi x/\ell}\right\}_{n=-\infty}^{\infty}$ on $(-\ell,\ell)$ are called the full real Fourier coefficients of f and are given by

$$\widehat{f}(n)=\frac{\left\langle f, e^{in\pi(\cdot)/\ell}\right\rangle}{\left\langle e^{in\pi(\cdot)/\ell}, e^{in\pi(\cdot)/\ell}\right\rangle}=\frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-in\pi x/\ell}dx$$

for $n=0,\pm1,\pm2,...$ The full complex Fourier series of f is

$$\sum_{n=-\infty}^{\infty}\widehat{f}(n)e^{in\pi x/\ell}.$$

Example 2. Let f be the 2-periodic function given on one period by

$$f(x)=e^{ax} \quad \text{for} \quad -1\le x<1,$$

where a is a nonzero real constant.

(a) Find the full complex Fourier series of f.

(b) Use part (a) to help write the full complex Fourier series of the 2-periodic function g given on one period by $g(x)=\cosh(x)$ for $-1\le x<1$.

(c) Use part (b) to help write the full real Fourier series of the function g in part (b).

Solution to example 1: Here $\ell = 1/2$ so

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = 2\int_{-1/2}^{1/2} \overbrace{\underbrace{(2x^4 - x^2)}_{\text{even}}\underbrace{\sin(2n\pi x)}_{\text{odd}}}^{\text{odd}}\,dx .$$

Since $\int_{-a}^{a} (\text{odd})\,dx = 0$, it follows that $b_n = 0$ for all $n \geq 1$.

(Note: When f is even, the full real sine coefficients of f are all zero.)

Also $\int_{-a}^{a} (\text{even})\,dx = 2\int_0^a (\text{even})\,dx$ so when $n \geq 1$,

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx = 2\int_{-1/2}^{1/2} (2x^4 - x^2)\cos(2n\pi x)\,dx$$

$$= 4\int_0^{1/2} (2x^4 - x^2)\cos(2n\pi x)\,dx .$$

Integrating by parts four times gives

$$a_n = \left.\frac{4(2x^4 - x^2)\sin(2n\pi x)}{2n\pi}\right|_0^{1/2} + \left.\frac{4(8x^3 - 2x)\cos(2n\pi x)}{(2n\pi)^2}\right|_0^{1/2}$$

$$- \left.\frac{4(24x^2 - 2)\sin(2n\pi x)}{(2n\pi)^3}\right|_0^{1/2} - \left.\frac{4(48x)\cos(2n\pi x)}{(2n\pi)^4}\right|_0^{1/2} + \frac{4}{(2n\pi)^4}\int_0^{1/2} 48\cos(2n\pi x)\,dx$$

$$= 0 + 0 - 0 - \frac{4(24)\cos(n\pi)}{(2n\pi)^4} + 0$$

$$= \frac{6(-1)^{n+1}}{(n\pi)^4} \qquad (n \geq 1) .$$

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx = \int_{-1/2}^{1/2}(2x^4-x^2)\,dx = 2\int_0^{1/2}(2x^4-x^2)\,dx = 2\left(\frac{2x^5}{5}-\frac{x^3}{3}\right)\Bigg|_0^{1/2} = \frac{-7}{120}$$

(a) Therefore the full real Fourier series of f is

$$a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{\ell}\right)+b_n\sin\left(\frac{n\pi x}{\ell}\right)\right] = \boxed{\frac{-7}{120}+\frac{6}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(2n\pi x)}{n^4}}$$

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(b) Assume that $f(x) = \frac{-7}{120}+\frac{6}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(2n\pi x)}{n^4}$ for all real x.

Taking $x = 1/2$ in this identity yields

$$2\left(\tfrac{1}{2}\right)^4-\left(\tfrac{1}{2}\right)^2 = f(1/2) = \frac{-7}{120}+\frac{6}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\overbrace{\cos(n\pi)}^{(-1)^n}}{n^4}.$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{6}\left(\frac{-7}{120}+\frac{1}{8}\right) = \boxed{\frac{\pi^4}{90}}$$

Alternatively, when we modeled diffusion in a closed narrow circular tube of length 2ℓ and encountered the eigenvalue problem

$$X''(x)+\lambda X(x)=0,\quad X(-\ell)=X(\ell) \text{ and } X'(-\ell)=X(\ell),$$

we could have expressed the eigenvalues/eigenfunctions in the form

$$\lambda_0 = 0,\quad X_0(x)=1$$

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2,\quad X_n(x) = e^{in\pi x/\ell},\quad \tilde{X}_n(x) = e^{-in\pi x/\ell}$$

$$(n=1,2,3,\dots)$$

The complete set of eigenfunctions for this problem,

$$\Phi = \left\{1, e^{i\pi x/\ell}, e^{-i\pi x/\ell}, e^{2i\pi x/\ell}, e^{-2i\pi x/\ell}, \dots\right\},$$

forms an orthogonal set on $(-\ell, \ell)$. The Fourier coefficients and series of f with respect to Φ are called the complex Fourier coefficients and complex Fourier series, respectively. (Refer to the handout "Examples for Sec. 5.2" for the formal definitions and the statement of example 2.)

Solution to example 2: Here $\ell = 1$, so

$$\hat{f}(n) = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\, e^{-in\pi x/\ell}\, dx = \frac{1}{2}\int_{-1}^{1} e^{ax}\cdot e^{-in\pi x}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{(a-in\pi)x}\, dx$$

$$= \frac{1}{2}\left.\frac{e^{(a-in\pi)x}}{a-in\pi}\right|_{-1}^{1}$$

$$= \frac{e^{a-in\pi} - e^{-a+in\pi}}{2(a-in\pi)}$$

$$e^{\pm in\pi} = \cos(n\pi) \pm i\sin(n\pi) = (-1)^n$$

$$= \frac{(-1)^n(e^{a} - e^{-a})}{2(a-in\pi)}$$

$$= \frac{(-1)^n \sinh(a)}{a-in\pi}$$

(a) Therefore $e^{ax} \sim \underbrace{\sum_{n=-\infty}^{\infty}\left[\frac{(-1)^n\sinh(a)}{a-in\pi}\right] e^{in\pi x}}$

Complex Fourier Series of $f(x) = e^{ax}$ on $(-1,1)$.

Note that $g(x) = \cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$ on $(-1,1)$.

$$e^{x} \sim \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n \sinh(1)}{1 - in\pi}\right] e^{in\pi x} = \sinh(1) \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{in\pi x}}{1-in\pi}$$

$$e^{-x} \sim \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n \sinh(-1)}{-1 - in\pi}\right] e^{in\pi x} = \sinh(1) \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{in\pi x}}{1+in\pi}$$

$$\therefore \quad g(x) \sim \frac{\sinh(1)}{2} \sum_{n=-\infty}^{\infty} (-1)^n \Bigg[\overbrace{\frac{1}{1-in\pi} + \frac{1}{1+in\pi}} \Bigg] e^{in\pi x}$$

$$\frac{1+in\pi + 1-in\pi}{(1-in\pi)(1+in\pi)} = \frac{2}{1+(n\pi)^2}$$

$$\sim \boxed{\sinh(1) \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{in\pi x}}{1+(n\pi)^2}}$$

Complex Fourier Series of $g(x) = \cosh(x)$ on $(-1,1)$.

(c)

$$g(x) \sim \sinh(1)\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{1+(k\pi)^2} \Big[\overbrace{e^{ik\pi x} + e^{-ik\pi x}}^{2\cos(k\pi x)} \Big] \right]$$

$$\sim \boxed{\sinh(1)\left[1 + \sum_{k=1}^{\infty} \frac{2(-1)^k \cos(k\pi x)}{1+(k\pi)^2} \right]}$$

Full Real Fourier Series of g