Mathematics 325
Supplementary Lecture Notes for Section 5.3
Orthogonality and General Fourier Series

Definition. For $n \geq 1$, let $C^n[a,b]$ denote the vector space of n times continuously differentiable complex-valued functions on the interval $[a,b]$. Let $C[a,b]$ denote the vector space of continuous complex-valued functions on $[a,b]$, and let V be a vector subspace of $C[a,b]$. A function T, defined on V, taking values in $C[a,b]$, and with the property that

$$T(c_1 f_1 + c_2 f_2) = c_1 T(f_1) + c_2 T(f_2)$$

for all numbers c_1 and c_2 and all f_1 and f_2 in V, is called a *linear operator* on V.

Example 1. The differential operator $T = -\frac{d^2}{dx^2}$ is a linear operator on $C^2[a,b]$.

Definition. A linear operator $T: V \to C[a,b]$ is called *symmetric* if $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all f and g in V.

Example 2. Show that the operator $T = -\frac{d^2}{dx^2}$ is symmetric on $V_D = \{f \in C^2[0,\pi] : f(0) = 0 = f(\pi)\}$.

Solution: Let f and g belong to V_D. Then two integrations by parts and use of the boundary conditions $f(0)=0=f(\pi)$ and $g(0)=0=g(\pi)$ show that

$$\langle Tf, g\rangle = \int_0^\pi Tf(x)\overline{g(x)}\,dx = \int_0^\pi -f''(x)\overline{g(x)}\,dx = \left[-f'(x)\overline{g(x)} + f(x)\overline{g'(x)}\right]\Big|_{x=0}^{\pi} - \int_0^\pi f(x)\overline{g''(x)}\,dx$$

$$= \int_0^\pi f(x)\left[\overline{-g''(x)}\right]dx.$$ Therefore $\langle Tf, g\rangle = \langle f, Tg\rangle$ so T is symmetric on V_D.

Homework A. Show that $T = -\frac{d^2}{dx^2}$ is symmetric on the following subspaces of $C[0,\pi]$.

1. $V_N = \{f \in C^2[0,\pi] : f'(0) = 0 = f'(\pi)\}$.
2. $V_P = \{f \in C^2[-\pi,\pi] : f(-\pi) = f(\pi), f'(-\pi) = f'(\pi)\}$.
3. $V_R = \{f \in C^2[0,\pi] : f'(0) - a_0 f(0) = 0 = f'(\pi) + a_\pi f(\pi)\}$.
   (Here a_0 and a_π are fixed real constants.)

Example 3. Show that the operator $Tf(x) = (1-x^2)f''(x) - xf'(x)$ is symmetric on the vector space $V_T = \{f \in C^2(-1,1) : f \text{ and } f' \text{ are bounded on } (-1,1)\}$ equipped with the inner product

(*) $$\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}(1-x^2)^{-1/2}\,dx.$$

Solution: Let f and g belong to V_T. Then $\langle Tf, g\rangle = \int_{-1}^{1}\left[(1-x^2)f''(x) - xf'(x)\right]\overline{g(x)}(1-x^2)^{-1/2}dx$

Solution (cont.):

$$\langle Tf, g\rangle = \int_{-1}^{1}\left[(1-x^2)^{1/2}f''(x) - x(1-x^2)^{-1/2}f'(x)\right]\overline{g(x)}\,dx = \int_{-1}^{1} \overbrace{\overline{g(x)}}^{v}\,\overbrace{\frac{d}{dx}\left[(1-x^2)^{1/2}f'(x)\right]dx}^{dv}$$

$$= (1-x^2)^{1/2}f'(x)\overline{g(x)}\Big|_{x=-1}^{1} - \int_{-1}^{1}(1-x^2)^{1/2}f'(x)\overline{g'(x)}\,dx.$$ Since $\lim\limits_{x\to 1^-}(1-x^2)^{-1/2}\underbrace{f'(x)\overline{g(x)}}_{\text{bounded}} = 0$ and

$\lim\limits_{x\to -1^+}(1-x^2)^{1/2}\underbrace{f'(x)\overline{g(x)}}_{\text{bounded}} = 0$ by the Squeeze Theorem, it follows that $(1-x^2)^{1/2}f'(x)\overline{g(x)}\Big|_{x=-1}^{1} = 0.$

Therefore $$\langle Tf, g\rangle = -\int_{-1}^{1}\overbrace{(1-x^2)^{1/2}\overline{g'(x)}}^{v}\,\overbrace{f'(x)dx}^{dv} = -(1-x^2)^{1/2}\overline{g'(x)}f(x)\Big|_{x=-1}^{1} + \int_{-1}^{1} f(x)\frac{d}{dx}\left[(1-x^2)^{1/2}\overline{g'(x)}\right]dx.$$

But an argument similar to the one above shows that $-(1-x^2)^{1/2}\overline{g'(x)}f(x)\Big|_{x=-1}^{1} = 0$ so

$$\langle Tf, g\rangle = \int_{-1}^{1} f(x)\left[(1-x^2)^{1/2}\overline{g''(x)} - x(1-x^2)^{-1/2}\overline{g'(x)}\right]dx = \int_{-1}^{1} f(x)\left[\overline{(1-x^2)g''(x) - xg'(x)}\right](1-x^2)^{-1/2}dx$$

and hence $\langle Tf, g\rangle = \langle f, Tg\rangle$. Consequently T is symmetric on V_T.

Definition. Let $T: V \to C[a,b]$ be a linear operator. If λ is a complex number and $f \neq 0$ is a function in V such that $Tf = \lambda f$ then λ is called an *eigenvalue* of T and f is called an *eigenfunction* of T.

Example 4. The operator $T = -\dfrac{d^2}{dx^2}$ on $V_D = \left\{f \in C^2[0,\pi] : f(0) = 0 = f(\pi)\right\}$ has eigenvalues $\lambda_n = n^2$ $(n = 1,2,3,\ldots)$ and corresponding eigenfunctions $f_n(x) = \sin(nx)$ $(n = 1,2,3,\ldots)$.

Theorem 2. Let $T: V \to C[a,b]$ be a symmetric operator. Then all the eigenvalues of T are real numbers.

Note: The proof of Theorem 2 will make use of the following properties of an inner product.

(1) For all f in V, $\langle f, f\rangle \geq 0$, with equality only if $f = 0$.
(2) For all f, g, and h in V, $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$.
(3) For all f and g in V and all complex numbers α, $\langle \alpha f, g\rangle = \alpha\langle f, g\rangle$.
(4) For all f and g in V, $\langle f, g\rangle = \overline{\langle g, f\rangle}$.

It is an easy consequence of (3) and (4) that $\langle f, \alpha g\rangle = \overline{\alpha}\langle f, g\rangle$.

Proof of Theorem 2: Let λ be an eigenvalue of T and let f be a nonzero function in V

Proof of Theorem 2 (cont.): such that $Tf = \lambda f$. Then $\lambda \langle f,f\rangle \overset{(3)}{=} \langle \lambda f, f\rangle =$
$\langle Tf, f\rangle \overset{\text{symmetry of } T}{=} \langle f, Tf\rangle = \langle f, \lambda f\rangle \overset{(3)\,\&\,(4)}{=} \bar{\lambda}\langle f,f\rangle$. Consequently,
$(\lambda-\bar{\lambda})\langle f,f\rangle = \lambda\langle f,f\rangle - \bar{\lambda}\langle f,f\rangle = 0$. But $f \neq 0$ so $\langle f,f\rangle > 0$ by (1).
Hence $\lambda - \bar{\lambda} = 0$ or equivalently $\lambda = \bar{\lambda}$. That is, λ is a real number.

Theorem 1. Let $T : V \to C[a,b]$ be a symmetric operator. If f_1 and f_2 are eigenfunctions of T corresponding to distinct eigenvalues λ_1 and λ_2 of T, then f_1 and f_2 are orthogonal on $[a,b]$.

Proof: Let λ_1 and λ_2 be distinct eigenvalues of T on V. That is, $\lambda_1 \neq \lambda_2$ and there exist nonzero functions f_1 and f_2 in V such that $Tf_1 = \lambda_1 f_1$ and $Tf_2 = \lambda_2 f_2$. Thus
$\lambda_1\langle f_1,f_2\rangle = \langle \lambda_1 f_1, f_2\rangle = \langle Tf_1, f_2\rangle \overset{\text{symmetry of } T}{=} \langle f_1, Tf_2\rangle = \langle f_1, \lambda_2 f_2\rangle = \bar{\lambda}_2\langle f_1,f_2\rangle$
$= \lambda_2\langle f_1,f_2\rangle$ by Theorem 2. Therefore $(\lambda_1-\lambda_2)\langle f_1,f_2\rangle = \lambda_1\langle f_1,f_2\rangle - \lambda_2\langle f_1,f_2\rangle = 0$.
But $\lambda_1 - \lambda_2 \neq 0$ so $\langle f_1,f_2\rangle = 0$. That is, f_1 and f_2 are orthogonal on $[a,b]$

Example 5. Let $Tf(x) = (1-x^2)f''(x) - xf'(x)$ be the operator on the vector space $V_T = \{f \in C^2(-1,1) : f \text{ and } f' \text{ are bounded on } (-1,1)\}$ equipped with the inner product

$$(*) \qquad \langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\left(1-x^2\right)^{-1/2} dx.$$

Show that all the eigenvalues of T are real numbers and the eigenfunctions of T corresponding to distinct eigenvalues are orthogonal on the interval $(-1,1)$ relative to the inner product (*).

Solution: Example 3 shows that the operator T is symmetric on V_T, equipped with the inner product (*). According to Theorem 2, all the eigenvalues of T are real numbers. By Theorem 1, eigenfunctions of T corresponding to distinct eigenvalues are orthogonal on $(-1,1)$ relative to the inner product (*).

Note: It can be shown that the symmetric operator T in Example 5 has eigenvalues $\lambda_n = -n^2$ $(n = 0,1,2,\ldots)$ and corresponding eigenfunctions that are the Tchebicheff polynomials: $f_n(x) = \cos\left(n\cos^{-1}(x)\right)$ $(n = 0,1,2,\ldots)$. The first three Tchebicheff polynomials are $f_0(x) = 1$, $f_1(x) = x$, and $f_2(x) = 2x^2 - 1$. The Tchebicheff polynomials are solutions to Tchebicheff's differential equation $(1-x^2)f''(x) - xf'(x) = \lambda f(x)$ on the interval $(-1,1)$ with $\lambda = \lambda_n = -n^2$.

Note on #1 of "Additional Problems for Section 5.3": Consider the operator

$$Tf(r) = \frac{1}{r}\frac{d}{dr}\left(rf'(r)\right) - \frac{n^2}{r^2}f(r) \qquad (0 < r \le 1)$$

on the domain

$$V_B = \left\{ f \in C^2(0,1] : f(1) = 0 \text{ and } f, f' \text{ are bounded on } (0,1] \right\}.$$

The inner product

$$\langle f, g \rangle = \int_0^1 f(r)\overline{g(r)}\, r\, dr$$

on V_B arises naturally from the inner product

$$\langle h, k \rangle = \int_0^{2\pi}\int_0^1 h(r,\theta)\overline{k(r,\theta)}\, r\, dr\, d\theta$$

for square-integrable functions h and k in the unit disk $D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ of the plane. The eigenvalue equation $Tf = \lambda f$ for this operator is equivalent to Bessel's equation of order n (cf. pp. 252 and 268 in Strauss):

$$\frac{1}{r}\frac{d}{dr}\left(rf'(r)\right) - \frac{n^2}{r^2}f(r) = \lambda f(r).$$

For applications of #1 in solving PDEs see Strauss, Section 10.2: Vibrations of a (Circular) Drumhead.

Theorem 3. Let $T = -\frac{d^2}{dx^2}$ be symmetric on a vector subspace V of $C^2[a,b]$ which is closed under the operation of complex conjugation of functions. If $f(b)f'(b) - f(a)f'(a) \le 0$ for all real-valued functions f in V then T has no negative eigenvalues.

Proof: Let λ be an eigenvalue of T on V and let f be an eigenfunction of T on V corresponding to λ; that is, f is a nonzero function in V such that $Tf = \lambda f$. This last condition is equivalent to

$$f''(x) + \lambda f(x) = 0 \quad \text{for all } x \text{ in } [a,b].$$

Take the complex conjugate of this identity, and use the fact that λ is a real number (cf. Theorem 2) to obtain $\overline{f''(x)} + \lambda\overline{f(x)} = 0$ for all x in $[a,b]$. Thus $T\overline{f} = \lambda\overline{f}$, so $\overline{f}$ is an eigenfunction of T on V corresponding to λ. Consequently, at least one of the functions

$$\phi = \operatorname{Re}(f) = \frac{1}{2}\left(f + \overline{f}\right) \quad \text{or} \quad \psi = \operatorname{Im}(f) = \frac{1}{2i}\left(f - \overline{f}\right)$$

is not the zero function and hence is a real-valued eigenfunction of T on V corresponding to λ. Suppose for the sake of argument that $\phi \ne 0$. Observe that

$$\lambda\langle\phi,\phi\rangle = \langle\lambda\phi,\phi\rangle = \langle T\phi,\phi\rangle = \int_a^b -\phi''(x)\overline{\phi(x)}\,dx = \int_a^b -\phi''(x)\phi(x)\,dx = -\phi'(b)\phi(b) + \phi'(a)\phi(a) + \int_a^b \left(\phi'(x)\right)^2 dx.$$

Since the last member of this identity is nonnegative and $\langle\phi,\phi\rangle$ is positive, it follows that $\lambda \ge 0$. Q.E.D.

Example 7. All the eigenvalues of $T = -\frac{d^2}{dx^2}$ on $V_D = \{f \in C^2[0,\pi] : f(0) = 0 = f(\pi)\}$ satisfy $\lambda \ge 0$.

Note: You will need to generalize Theorem 3 and its proof in order to work #1(c) on "Additional Problems for Section 5.3".

Math 325
Section 5.3 Supplementary Problems

1. (a) Let n be a nonnegative integer. Show that the operator T given by

$$Tf(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2}f(r) \qquad (0 < r \le 1)$$

is symmetric on the vector space

$$V_B = \left\{ f \in C^2(0,1] : f(1) = 0,\ f \text{ and } f' \text{ bounded on } (0,1] \right\}$$

equipped with the inner product

(*) $$\langle f, g\rangle = \int_0^1 f(r)\overline{g(r)}r\,dr.$$

(b) Show that the eigenvalues of T on V_B real numbers.

(c) Are the eigenvalues of T on V_B positive? Justify your answer.

(d) Are the eigenfunctions of T on V_B, corresponding to distinct eigenvalues, orthogonal on $(0,1)$ relative to the inner product (*)? Justify your answer.

2. Use separation of variables to solve the variable density vibrating string problem:

$$\frac{1}{(1+x)^2}u_{tt} - u_{xx} = 0 \quad \text{for } 0 < x < 1,\ 0 < t < \infty,$$
$$u(0,t) = 0 \text{ and } u(1,t) = 0 \quad \text{for } 0 \le t < \infty,$$
$$u(x,0) = x(1-x)\sqrt{1+x} \text{ and } u_t(x,0) = 0 \quad \text{for } 0 \le x \le 1.$$

Hints on 2: (a) Show that the operator T given by $Tf(x) = -(1+x)^2 f''(x)$ is symmetric on $V_D = \left\{ f \in C^2[0,1] : f(0) = 0 = f(1)\right\}$, equipped with the inner product $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}(1+x)^{-2}\,dx$.

Conclude that all the eigenvalues λ of the problem $X''(x) + \frac{\lambda}{(1+x)^2}X(x) = 0,\ X(0) = 0 = X(1)$ are real.

(b) Show that (nearly) all solutions to $X''(x) + \frac{\lambda}{(1+x)^2}X(x) = 0$ on $(0,1)$ are of the form $X(x) = (1+x)^a$ where a is an appropriately chosen (possibly complex) constant. Explicitly, show that the general solution is:

$$X(x) = (1+x)^{1/2}\left[c_1(1+x)^{\frac{\sqrt{1-4\lambda}}{2}} + c_2(1+x)^{\frac{-\sqrt{1-4\lambda}}{2}}\right] \quad \text{if } 1-4\lambda > 0,$$

$$X(x) = (1+x)^{1/2}\left[c_1 + c_2\ln(1+x)\right] \quad \text{if } 1-4\lambda = 0,$$

$$X(x) = (1+x)^{1/2}\left[c_1\cos\left(\frac{\sqrt{4\lambda-1}}{2}\ln(1+x)\right) + c_2\sin\left(\frac{\sqrt{4\lambda-1}}{2}\ln(1+x)\right)\right] \quad \text{if } 1-4\lambda < 0.$$