Mathematics 325
Lecture Notes for Section 6.1 (Lecture)
Laplace's Equation

Laplace's equation is $\nabla^2 u = 0$. In particular, this is in :

One dimension: $\dfrac{d^2u}{dx^2} = 0$ with solution $u(x) = ax + b$ where a and b are arbitrary constants;

Two dimensions: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ with solutions being much more complex. Some examples are:

$u(x,y) = ax + by + c$ $\quad$ (a, b, c constants)

$u(x,y) = xy$

$u(x,y) = x^2 - y^2$

2D "Fundamental Solution" $\rightarrow$ $u(x,y) = \ln\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)$ $\quad$ ($(x,y) \neq (0,0)$)

$\nabla^2 u = \delta$ $\quad$ Three dimensions: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ with solutions being even more complex.

3D "Fundamental Soln" $\nabla^2 u = \delta$

$u(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ if $(x,y,z) \neq (0,0,0)$

Homework A. Find all solutions of the form

$$u(x,y) = ax^2 + bxy + cy^2 + dx + ey + f \qquad (a, b, \ldots, f \text{ constants})$$

to the two-dimensional Laplace equation.

(*) Soln: $u(x,y) = a(x^2 - y^2) + bxy + dx + ey + f$

The inhomogeneous Laplace equation is called *Poisson's equation*: $\nabla^2 u = f$.

Definition. If u is a solution to Laplace's equation $\nabla^2 u = 0$ in a region D, then u is called a *harmonic function* in D.

Applications of the Laplace/Poisson Equation.

1. Stationary (or Steady-State) Diffusions and Waves.

If $u = u(\mathbf{x}, t)$ is a solution to the wave equation $u_{tt} - c^2 \nabla^2 u = 0$ and u is independent of t then $0 = u_t = u_{tt}$, so u solves Laplace's equation $\nabla^2 u = 0$. Such a solution u is called a "standing wave". (Similar statements hold for time-independent solutions of the diffusion equation $u_t - k\nabla^2 u = 0$.)

2. Conservative Field Theory (e.g. electrostatics, irrotational hydrostatics, gravitation).

A field $\mathbf{F}$ is *conservative* in a region D of $\mathbb{R}^3$ if there exists a C^1 real-valued function ϕ such that $\mathbf{F} = -\nabla\phi$. In such a case, the function ϕ is called a *potential* for $\mathbf{F}$ in D, and if $\mathbf{F}$ is divergence-free then

$$0 = \nabla \cdot \mathbf{F} = \nabla \cdot (-\nabla\phi) = -\nabla^2 \phi \quad \text{in} \quad D.$$

That is, the potential ϕ is harmonic in D. Laplace exploited this idea to systematically solve problems in celestial mechanics.

3. Analytic Function Theory.

Definition. A complex-valued function f is *analytic* in a region D of the complex plane provided f has a power series representation

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

in some nonempty open disk $|z-z_0|<r$ about each point z_0 in D.

FACT (cf. exercise 1 in Section 6.1). The function f is analytic in a region D if and only if

$$f(x+iy)=u(x,y)+iv(x,y) \qquad (x+iy=z \text{ in } D)$$

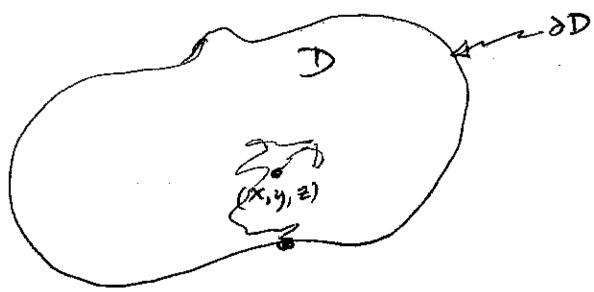
for some harmonic real-valued functions u and v in D which satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Note: As a consequence of FACT, any solution to Laplace's equation $\nabla^2 u=0$ in D is actually a C^∞-function in D. Another proof that any harmonic function is C^∞ is given in the text on page 163.

4. Brownian Motion.

Let $X(x,y,z)$ denote the average first exit time of a particle undergoing Brownian motion in a bounded region D with initial position (x,y,z).



Then X is harmonic in D:

$$X_{xx}+X_{yy}+X_{zz}=0.$$

This and related results are useful in the mathematical modeling of financial markets.

FACT. Harmonic functions satisfy a maximum/minimum principle (cf. pp. 148-9). This leads to a uniqueness result (cf. pp. 149-150) for solutions to Dirichlet problems involving Laplace's equation:

$$\begin{cases} \nabla^2 u = f & \text{in } D, \\ u = g & \text{on } \partial D. \end{cases}$$

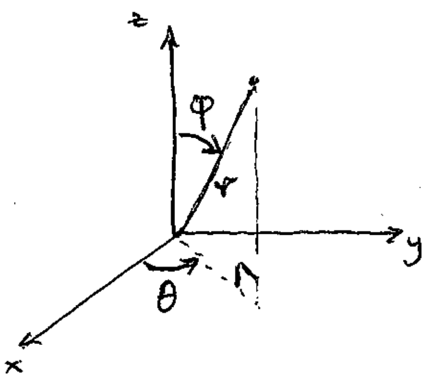
See exercise 11 on page 168 for Robin problems and exercise 12(b) on page 168 and exercise 2 on page 174 for Neumann problems involving Laplace's equation.

The Laplacian operator $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ exhibits rotational and translational invariance. In particular, $\nabla^2=\nabla_R^2$ where R is any rotation. (For proofs, see page 150 for two-dimensional rotations and page 152 for three-dimensional rotations.) This is why the Laplacian often appears when modeling isotropic phenomena (i.e. those with no preferred direction).

The Laplacian operator has the following representations.

	Cartesian Coordinates	Polar or Spherical Polar Coordinates
2-D:	$\nabla^2 u = u_{xx} + u_{yy}$	$\nabla^2 u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$
3-D:	$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$	$\nabla^2 u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\left[\cot(\phi)\dfrac{\partial u}{\partial \phi} + \dfrac{\partial^2 u}{\partial \phi^2}\right] + \dfrac{1}{r^2\sin^2(\phi)}\dfrac{\partial^2 u}{\partial \theta^2} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2\sin(\phi)}\dfrac{\partial}{\partial \phi}\left(\sin(\phi)\dfrac{\partial u}{\partial \phi}\right) + \dfrac{1}{r^2\sin^2(\phi)}\dfrac{\partial^2 u}{\partial \theta^2}$

Beware: The notation that I use for spherical polar coordinates in three dimensions is standard in mathematics, but it differs from that commonly used in physics and our PDE text. Consequently, the representation for the Laplacian I give above differs from Strauss (cf. p. 153). For clarity, the diagram below gives the notation I use for three-dimensional spherical polar coordinates.



Note that I denote the longitude by θ and the colatitude by ϕ.

Example 1. (#5 on p. 154) Solve $u_{xx} + u_{yy} = 1$ if $x^2 + y^2 < a^2$ subject to the boundary condition $u = 0$ if $x^2 + y^2 = a^2$.

Soln: Because the PDE, region, and B.C. are invariant under rotation, we expect the solution to have the form $u(r,\theta) = u(r)$, independent of θ. (Such solutions are called "radial".) Writing the PDE in polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\cancelto{0}{\frac{\partial^2 u}{\partial \theta^2}} = 1 \quad \Rightarrow \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 1 \quad \Rightarrow \quad \frac{d}{dr}\left(r\frac{du}{dr}\right) = r$$

$$\Rightarrow \quad r\frac{du}{dr} = \frac{r^2}{2} + c_1 \quad \Rightarrow \quad \frac{du}{dr} = \frac{r}{2} + \frac{c_1}{r} \quad \Rightarrow \quad u = \frac{r^2}{4} + c_1 \ln(r) + c_2$$

In order for the solution to be defined at the origin, $c_1 = 0$. I.e. $u(r) = \frac{r^2}{4} + c_2$.

(OVER)

Apply the B.C. $u(a)=0$ to get $\quad 0 = \frac{a^2}{4} + c_2 \Rightarrow c_2 = -\frac{a^2}{4}$.

Thus $\boxed{u(r,\theta) = \frac{r^2-a^2}{4}}$ or equivalently $\boxed{u(x,y) = \frac{x^2+y^2-a^2}{4}}$

Example 2. (#8 on p. 154) Solve $\nabla^2 u = 1$ in the spherical shell $a<r<b$, given that $u=0$ on $r=a$ and $\frac{\partial u}{\partial r}=0$ on $r=b$.

Soln: Arguing as before on the radial symmetry of the PDE, region, and B.C.'s, we expect a solution of the form $u(r,\theta,\varphi) = u(r)$, independent of φ and θ. Writing the PDE in spherical coordinates yields

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\,\cancelto{0}{\frac{\partial u}{\partial\varphi}}\right) + \frac{1}{r^2\sin^2\varphi}\cancelto{0}{\frac{\partial^2 u}{\partial\theta^2}} = 1$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = r^2 \Rightarrow r^2\frac{\partial u}{\partial r} = \frac{r^3}{3} + c_1 \Rightarrow \frac{\partial u}{\partial r} = \frac{r}{3} + \frac{c_1}{r^2}$$

Apply the BC: $\frac{\partial u}{\partial r}=0$ when $r=b$ to get $\quad 0 = \frac{b}{3} + \frac{c_1}{b^2} \Rightarrow c_1 = -\frac{b^3}{3}$

$$\therefore \frac{\partial u}{\partial r} = \frac{r}{3} - \frac{b^3}{3r^2}$$

$$\Rightarrow u = \frac{r^2}{6} + \frac{b^3}{3r} + c_2$$

Apply the BC: $u=0$ on $r=a$ to get $\quad 0 = \frac{a^2}{6} + \frac{b^3}{3a} + c_2 \Rightarrow c_2 = \frac{-a^2}{6} - \frac{b^3}{3a}$

$$\therefore \boxed{u(r)} = \frac{r^2}{6} + \frac{b^3}{3r} - \frac{a^2}{6} - \frac{b^3}{3a} = \boxed{\frac{1}{6}(r^2-a^2) + \frac{b^3}{3}\left(\frac{1}{r}-\frac{1}{a}\right)}$$

or equivalently $\boxed{u(x,y,z) = \frac{x^2+y^2+z^2-a^2}{6} + \frac{b^3}{3}\left(\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{1}{a}\right)}$

Answer: Yes. Consider the energy ~~[illegible]~~ $E = \int |\vec{\nabla} w|^2\, dV$ of $w = u - v$ where u & v are two solutions of problem.

Question: Is this the only solution?