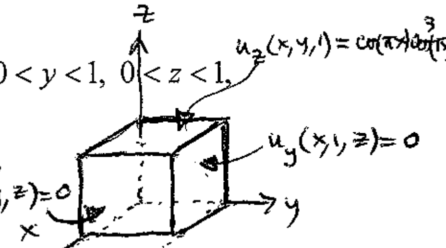
Mathematics 325
Lecture Notes for Section 6.2
Laplace's Equation in Rectangles and Cubes

We have already seen how to solve Laplace's equation in a rectangle. (See the examples from Sections 4.1 and 5.1.) We will now look at the case of cubes.

Example 1. (Similar to #6 on page 158) Solve $\nabla^2 u = 0$ in the cube $0 < x < 1,\ 0 < y < 1,\ 0 < z < 1,$ subject to the inhomogeneous Neumann boundary condition

$$u_z(x, y, 1) = \cos(\pi x)\cos^3(\pi y) \quad \text{if } 0 \le x \le 1,\ 0 \le y \le 1,$$

and homogeneous Neumann boundary conditions on the other five faces.



Solution. Since we are solving a PDE on a bounded region, we use separation of variables. We seek nontrivial solutions to the homogeneous part of the problem above of the form $u(x, y, z) = X(x)Y(y)Z(z)$. Differentiating and substituting this functional form for the solution into the PDE $u_{xx} + u_{yy} + u_{zz} = 0$ gives

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$$

and dividing by $X(x)Y(y)Z(z)$ we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0 .$$

Separating variables successively yields

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = \text{constant} = \lambda$$

and then

$$-\frac{Y''(y)}{Y(y)} = \frac{Z''(z)}{Z(z)} - \lambda = \text{constant} = \mu .$$

Thus we arrive at the coupled system of three ODEs:

$$X''(x) + \lambda X(x) = 0,$$
$$Y''(y) + \mu Y(y) = 0,$$
$$Z''(z) - (\lambda + \mu)Z(z) = 0,$$

where λ and μ are independent constants.

The homogeneous Neumann boundary condition on the face $x = 1$ (the front face) of the cube is $u_x(1, y, z) = 0$. Substituting the functional form $u(x, y, z) = X(x)Y(y)Z(z)$ into this boundary condition leads to

$$X'(1)Y(y)Z(z) = 0$$

for all $0 \le y \le 1,\ 0 \le z \le 1$. Since we seek nontrivial solutions, it follows that

$$X'(1) = 0.$$

Similar reasoning for the homogeneous Neumann conditions on the other four faces leads to:

$$X''(x) + \lambda X(x) = 0,\quad X'(0) = 0 = X'(1),$$
$$Y''(y) + \mu Y(y) = 0,\quad Y'(0) = 0 = Y'(1),$$
$$Z''(z) - (\lambda + \mu)Z(z) = 0,\quad Z'(0) = 0.$$

Eigenvalue problems.

Notice that the first two lines above contain eigenvalue problems but there is just one boundary condition for the second order ODE in z in the last line, so it is not an eigenvalue problem.

From previous work in Section 4.2, we know that the eigenvalues and eigenfunctions for the problems in the first two lines are, respectively,

$$\lambda_\ell = (\ell\pi)^2, \quad X_\ell(x) = \cos(\ell\pi x) \qquad (\ell = 0, 1, 2, \ldots)$$

and

$$\mu_m = (m\pi)^2, \quad Y_m(y) = \cos(m\pi y) \qquad (m = 0, 1, 2, \ldots).$$

Substituting $\lambda_l + \mu_m = (l\pi)^2 + (m\pi)^2$ into the differential equation in the third line, we get

$$Z_{\ell,m}''(z) - \pi^2(\ell^2 + m^2) Z_{\ell,m}(z) = 0, \quad Z_{\ell,m}'(0) = 0.$$

Hence

$$Z_{\ell,m}(z) = A_{\ell,m} \cosh\left(\pi z \sqrt{\ell^2+m^2}\right)$$

where $A_{l,m}$ is an arbitrary constant. Therefore a formal solution to the homogeneous part of the problem is

$$u(x,y,z) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} A_{\ell,m} \cosh\left(\pi z \sqrt{\ell^2+m^2}\right) \cos(\ell\pi x) \cos(m\pi y).$$

If possible, we want to choose the constants so that the inhomogeneous boundary condition is met:

$$\cos(\pi x)\cos^3(\pi y) = u_z(x,y,1) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \pi\sqrt{\ell^2+m^2}\, A_{\ell,m} \sinh\left(\pi\sqrt{\ell^2+m^2}\right) \cos(\ell\pi x)\cos(m\pi y)$$

for all $0 \le x \le 1$, $0 \le y \le 1$. Using the identity $\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta)$, this becomes

$$\frac{3}{4}\cos(\pi x)\cos(\pi y) + \frac{1}{4}\cos(\pi x)\cos(3\pi y) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \pi\sqrt{\ell^2+m^2}\, A_{\ell,m} \sinh\left(\pi\sqrt{\ell^2+m^2}\right) \cos(\ell\pi x)\cos(m\pi y)$$

for all $0 \le x \le 1$, $0 \le y \le 1$. By inspection,

$$\frac{3}{4} = \pi\sqrt{2}\, A_{1,1} \sinh(\pi\sqrt{2}),$$

$$\frac{1}{4} = \pi\sqrt{10}\, A_{1,3} \sinh(\pi\sqrt{10}),$$

$$A_{0,0} \text{ is arbitrary},$$

and all other $A_{l,m} = 0$. Therefore

$$A_{1,1} = \frac{3}{4\pi\sqrt{2}\sinh(\pi\sqrt{2})} \quad \text{and} \quad A_{1,3} = \frac{1}{4\pi\sqrt{10}\sinh(\pi\sqrt{10})}$$

so a solution is

$$u(x,y,z) = A_{0,0} + \frac{3\cos(\pi x)\cos(\pi y)\cosh(\pi z\sqrt{2})}{4\pi\sqrt{2}\sinh(\pi\sqrt{2})} + \frac{\cos(\pi x)\cos(3\pi y)\cosh(\pi z\sqrt{10})}{4\pi\sqrt{10}\sinh(\pi\sqrt{10})}$$

where $A_{0,0}$ is an arbitrary constant. In particular, note that there is not a unique solution to this problem.

Example 2. (#6 on page 158) Solve $\nabla^2 u = 0$ in the cube $0 < x < 1,\ 0 < y < 1,\ 0 < z < 1$, subject to the inhomogeneous Neumann boundary condition

$$u_z(x,y,1) = g(x,y) \quad \text{if } 0 \le x \le 1,\ 0 \le y \le 1,$$

and homogeneous Neumann boundary conditions on the other five faces. Here $g = g(x,y)$ is a square-integrable function with zero average on the unit square S: $0 \le x \le 1,\ 0 \le y \le 1$.

Solution. Proceeding as in Example 1, we find that

$$u(x,y,z) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} A_{l,m} \cosh\left(\pi z\sqrt{l^2+m^2}\right)\cos(l\pi x)\cos(m\pi y)$$

where the constants $A_{l,m}$ are chosen so as to satisfy the inhomogeneous Neumann condition:

$$(*) \qquad g(x,y) = u_z(x,y,1) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \pi\sqrt{l^2+m^2}\,A_{l,m} \sinh\left(\pi\sqrt{l^2+m^2}\right)\cos(l\pi x)\cos(m\pi y)$$

for all $0 \le x \le 1,\ 0 \le y \le 1$.

It is not hard to see that the functions $\psi_{l,m}(x,y) = \cos(l\pi x)\cos(m\pi y)$ $(l = 0,1,2,\ldots$ and $m = 0,1,2,\ldots)$ form an orthogonal system on the unit square S. This can be shown directly by verifying

$$\langle \psi_{l,m}, \psi_{l',m'} \rangle = \int_0^1\int_0^1 \psi_{l,m}(x,y)\overline{\psi_{l',m'}(x,y)}\,dxdy = 0 \quad \text{if } (l,m) \ne (l',m').$$

However, a faster way to show orthogonality is to observe that $\{\psi_{l,m}\}_{l,m=0}^{\infty}$ consists of eigenfunctions of the symmetric operator $-\nabla^2 = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ on

$$V = \left\{\psi \in C^2(S):\ \psi_x(1,y) = 0 = \psi_x(0,y) \text{ for all } 0 \le y \le 1 \text{ and } \psi_y(x,0) = 0 = \psi_y(x,1) \text{ for all } 0 \le x \le 1\right\}.$$

Using general Fourier series, we "know" that

$$(+) \qquad g(x,y) \sim \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} B_{l,m}\psi_{l,m}(x,y) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} B_{l,m}\cos(l\pi x)\cos(m\pi y)$$

where the coefficients $B_{l,m}$ are the Fourier coefficients of g with respect to the orthogonal system $\{\psi_{l,m}\}_{l,m=0}^{\infty}$ on S:

$$(\#) \qquad B_{l,m} = \frac{\langle g, \psi_{l,m}\rangle}{\langle \psi_{l,m}, \psi_{l,m}\rangle} = \frac{\int_0^1\int_0^1 g(x,y)\cos(l\pi x)\cos(m\pi y)\,dxdy}{\int_0^1\int_0^1 \cos^2(l\pi x)\cos^2(m\pi y)\,dxdy}.$$

Therefore, comparing (*) and (+) we see that

$$(\%) \qquad B_{l,m} = \pi\sqrt{l^2+m^2}\,A_{l,m}\sinh\left(\pi\sqrt{l^2+m^2}\right)$$

where $B_{l,m}$ is given by (#). Note that if $(l,m) = (0,0)$ then the right member of (%) is zero for arbitrary $A_{0,0}$. Hence, we must have $0 = B_{0,0} = \int_0^1\int_0^1 g(x,y)\,dxdy$ in order for a solution to exist for Example 2. That is, g must have average value zero in the unit square S.

To summarize, a solution to Example 2 is given by

$$u(x,y,z) = A_{0,0} + \sum_{\substack{l=0 \\ (l,m)\neq(0,0)}}^{\infty} \sum_{m=0}^{\infty} \frac{B_{l,m} \cosh\left(\pi z\sqrt{l^2+m^2}\right)\cos(l\pi x)\cos(m\pi y)}{\pi\sqrt{l^2+m^2}\,\sinh\left(\pi\sqrt{l^2+m^2}\right)}$$

provided g has average value zero in the unit square S, $A_{0,0}$ is an arbitrary constant, and the coefficients $B_{l,m}$ are given by (#).