Mathematics 325 Exam I Name: Dr. Grow

1.(25 pts.) Solve the partial differential equation $(1-x^2)u_x + xyu_y = 0$ subject to $u(0,y) = y^4$ for all $-\infty < y < \infty$. In which region in the xy-plane is the solution uniquely defined?

The characteristics of the p.d.e. satisfy

4 pts. to here.

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} = \frac{xy}{1-x^2},$$

so separating variables gives

10 pts. to here.

$$\ln|y| = \int \frac{dy}{y} = \int \frac{x\,dx}{1-x^2} = -\frac{1}{2}\int \frac{-2x\,dx}{1-x^2} = -\frac{1}{2}\ln|1-x^2| + c_1 .$$

Rearranging we have

$$2\ln|y| + \ln|1-x^2| = c \quad \text{or} \quad \ln|y^2(1-x^2)| = c \qquad (c = 2c_1)$$

12 pts. to here

so $y^2(1-x^2) = A$ (where $A = \pm e^c$) are the characteristic curves. Along such a curve the value of $u = u(x,y)$ is constant. Thus, along a characteristic curve

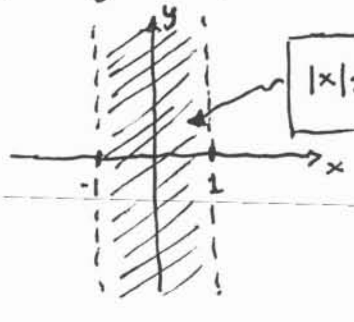
$$u(x,y) = u\left(x, \frac{A}{\pm\sqrt{|1-x^2|}}\right) = u\left(0, \frac{A}{\pm\sqrt{1-0}}\right) = f(A).$$

17 pts. to here.

Thus the general solution to the p.d.e. is $u(x,y) = f(y^2(1-x^2))$ where f is a differentiable function of a single real variable. We need to find f so that the initial condition is satisfied: $y^4 = u(0,y) = f(y^2(1-0^2)) = f(y^2)$ for all real y. Therefore $f(w) = w^2$ for all $w \geq 0$. Consequently, the solution of the I.V.P. is $u(x,y) = (y^2(1-x^2))^2 = \boxed{y^4(1-x^2)^2}$. Since $f(w)$ is determined uniquely only for nonnegative arguments: $w \geq 0$, the solution $u(x,y) = y^4(1-x^2)^2$ is uniquely determined only for $y^2(1-x^2) \geq 0$; i.e. $|x| \leq$

22 pts. to here

pts. to here.



$\boxed{|x| \leq 1 \text{ is the region where } u \text{ is uniquely determined.}}$

2.(25 pts.) Find the general solution of $u_x + 2u_y + 7(2x-y)u = 7(2x-y)(x+2y)$ in the xy-plane.

We use the change-of-coordinate method. Let

$$\xi = 2x - y$$
$$\eta = x + 2y$$

1 pts. to here.

Then the chain rule for derivatives gives the following operator equivalences:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = -\frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta}.$$

6 pts. to here.

Substituting these expressions in the p.d.e. yields

$$\left(2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)u + 2\left(-\frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta}\right)u + 7\xi u = 7\xi\eta$$

or

$$5\frac{\partial u}{\partial \eta} + 7\xi u = 7\xi\eta$$

so

$$\frac{\partial u}{\partial \eta} + \frac{7\xi}{5}u = \frac{7\xi\eta}{5}. \quad \left(\begin{array}{l}1^{st}\text{-order linear ODE in } \eta \\ \text{with } \xi \text{ as a parameter}\end{array}\right)$$

1 pts. to here

3 pts. to here

An integrating factor is $e^{\int p(\eta)d\eta} = e^{\int \frac{7\xi}{5}d\eta} = e^{\frac{7\xi\eta}{5}}$. Multiplying through the DE above by the integrating factor yields an exact expression on the left hand side.

15 pts. to here.

$$e^{\frac{7\xi\eta}{5}}\frac{\partial u}{\partial \eta} + \frac{7\xi}{5}e^{\frac{7\xi\eta}{5}}u = \frac{7\xi\eta}{5}e^{\frac{7\xi\eta}{5}}$$

or

$$\frac{\partial}{\partial \eta}\left[e^{\frac{7\xi\eta}{5}}u\right] = \frac{7\xi\eta}{5}e^{\frac{7\xi\eta}{5}}.$$

Integrating both sides with respect to η holding ξ fixed gives

$$e^{\frac{7\xi\eta}{5}}u = \int \underbrace{\frac{7\xi\eta}{5}}_{U}\underbrace{e^{\frac{7\xi\eta}{5}}d\eta}_{dV} = \frac{7\xi\eta}{5}\cdot\frac{5}{7\xi}e^{\frac{7\xi\eta}{5}} - \int \frac{\cancel{5}}{\cancel{7\xi}}e^{\frac{7\xi\eta}{5}}\frac{\cancel{7\xi}}{\cancel{5}}d\eta$$

so $e^{\frac{7\xi\eta}{5}}u = \eta e^{\frac{7\xi\eta}{5}} - \frac{5}{7\xi}e^{\frac{7\xi\eta}{5}} + c(\xi) \Rightarrow u = \eta - \frac{5}{7\xi} + c(\xi)e^{-\frac{7\xi\eta}{5}}.$

22 pts. to here.

As a function of x and y,

$$\boxed{u(x,y) = x + 2y - \frac{5}{7(2x-y)} + f(2x-y)e^{-\frac{7(2x-y)(x+2y)}{5}}}$$

25 pts. to here.

where f is a differentiable function of a single real variable.

3.(25 pts.) (a) Classify the partial differential equation $u_{tt} - c^2 u_{xx} = 0$ as elliptic, parabolic, or hyperbolic.

(b) Find the general solution of $u_{tt} - c^2 u_{xx} = 0$ in the xt-plane.

(c) Find the solution of $u_{tt} - c^2 u_{xx} = 0$ in the xt-plane satisfying $u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$ for all $-\infty < x < \infty$.

3pts. (a) $B^2 - 4AC = 0^2 - 4(1)(-c^2) = 4c^2 > 0$. The p.d.e. is $\boxed{\text{hyperbolic}}$ (if $c \neq 0$).

5 pts. to here. (b) The p.d.e. is $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$ in factored form for the operator. This suggests the change-of-coordinates:

$$\xi = ct + x$$

7 pts. to here.

$$\eta = ct - x .$$

The chain rule for derivatives implies the operator identities:

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}$$

9 pts. to here.

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$$

and hence $\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} = 2c\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} = 2c\frac{\partial}{\partial \xi}$. Therefore the p.d.e. is equivalent to

$$\left(2c\frac{\partial}{\partial \eta}\right)\left(2c\frac{\partial}{\partial \xi}\right)u = 0 \qquad \text{or} \qquad \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0.$$

11 pts. to here.

Integrating with respect to η holding ξ fixed yields $\frac{\partial u}{\partial \xi} = c_1(\xi)$, and then

13 pts. to here. integrating with respect to ξ holding η fixed leads to $u = \underbrace{\int c_1(\xi)\,d\xi}_{f(\xi)} + \underbrace{c_2(\eta)}_{g(\eta)}$.

15 pts. to here. As a function of x and t, $\boxed{u(x,t) = f(ct+x) + g(ct-x)}$ where f and g are any twice-differentiable functions of a single real variable.

(c) Applying the initial conditions we have

16 pts to here.

① $\varphi(x) = u(x,0) = f(x) + g(-x)$

and

17 pts. to here.

② $\psi(x) = u_t(x,0) = cf'(x) + cg'(-x)$

for all $-\infty < x < \infty$. Differentiating ① and multiplying by c gives

①' $c\varphi'(x) = cf'(x) - cg'(-x)$.

(OVER)

Adding ② and ①′ gives $\psi(x) + c\varphi'(x) = 2cf'(x)$, and solving for f gives

20 pts. to here.

$$f(x) = \int f'(x)\,dx = \frac{1}{2c}\int\left[\psi(x) + c\varphi'(x)\right]dx = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(\xi)d\xi + c_1$$

On the other hand, subtracting ①′ from ② gives

$$\psi(x) - c\varphi'(x) = 2cg'(-x).$$

Then $\psi(-x) - c\varphi'(-x) = 2cg'(x)$ so solving for g gives

23 pts. to here.

$$g(x) = \int g'(x)\,dx = \frac{1}{2c}\int\left[\psi(-x) - c\varphi'(-x)\right]dx = \frac{1}{2}\varphi(-x) + \frac{1}{2c}\int_0^x \psi(-\xi)d\xi$$

$$= \frac{1}{2}\varphi(-x) + \frac{1}{2c}\int_{-x}^0 \psi(\xi)d\xi + c_2$$

Therefore $u(x,t) = f(x+ct) + g(ct-x)$

$$= \frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi(\xi)d\xi + c_1 + \frac{1}{2}\varphi(-ct+x) + \frac{1}{2c}\int_{-ct+x}^0 \psi(\xi)d\xi.$$

25 pts. to here.

d'Alembert's formula

$$\boxed{u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)d\xi}.$$

Must ha

(Note: $c_1 + c_2 = 0$ since $u(x,0) = \frac{1}{2}[\varphi(x) + \varphi(x)] + c_1 + c_2 = \varphi(x$

for all real x.)

4.(25 pts.) Let $u = u(x,y,z,t)$ denote the temperature at time $t \geq 0$ at each point (x,y,z) of a homogeneous body occupying the spherical region $B = \{(x,y,z):\ x^2+y^2+z^2 \leq 25\}$. The body is completely insulated and the initial temperature at each point is equal to its distance from the center of B.

(a) Write (without proof or derivation) the partial differential equation and the complete initial/boundary conditions that govern the temperature function.

(b) Use Gauss' divergence theorem to help show that the heat energy $H(t) = \iiint_B c\rho u(x,y,z,t)\,dxdydz$ of the body at time t is actually a constant function of time. (Here c and ρ denote the constant specific heat and density, respectively, of the material in B.)

(c) Compute the constant steady-state temperature that the body reaches after a long time.

9 (a)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{k_0}{c\rho}\nabla^2 u = 0 & \text{if } 0 < t < \infty \text{ and } x^2+y^2+z^2 < 25 \\ \nabla u \cdot \vec{n} = 0 & \text{if } 0 \leq t < \infty \text{ and } x^2+y^2+z^2 = 25 \\ u(x,y,z,0) = \sqrt{x^2+y^2+z^2} & \text{if } x^2+y^2+z^2 \leq 25 \end{cases}$$

outward-pointing normal to ∂B

3 pts. for each line (9 total pts.)

8 (b)

$$\frac{dH}{dt} = \frac{d}{dt}\iiint_B c\rho u(x,y,z,t)\,dxdydz = \iiint_B c\rho u_t(x,y,z,t)\,dxdydz$$

2 pts.

$$= \iiint_B k_0 \nabla^2 u(x,y,z,t)\,dxdydz \overset{\text{Gauss' Div. Thm.}}{=} \iint_{\partial B} k_0 \underbrace{\nabla u\cdot\vec{n}}_{0 \text{ on } \partial B}\,dS = 0.$$

(5 pts.)

Therefore $H(t) = H(0)$ for all $t \geq 0$.

(1 pts.)

8 (c)

$$H(0) = \lim_{t\to\infty} H(t) = \lim_{t\to\infty}\iiint_B c\rho u(x,y,z,t)\,dxdydz = \iiint_B c\rho\Big[\overbrace{\lim_{t\to\infty} u(x,y,z,t)}^{U}\Big]dxdydz$$

3 pts. to here.

$$= \iiint_B c\rho U\,dxdydz = c\rho U \operatorname{vol}(B) = c\rho U\cdot\tfrac{4}{3}\pi(5)^3.$$ On the other hand,

$$H(0) = \iiint_B c\rho \overbrace{u(x,y,z,0)}^{r}\,\overbrace{dxdydz}^{r^2\sin\varphi\, dr\, d\varphi\, d\theta} = \int_0^{2\pi}\int_0^{\pi}\int_0^{5} c\rho r\cdot r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\varphi\,d\varphi\right)\left(\int_0^5 c\rho r^3\,dr\right) = (2\pi)(2)\left(\frac{c\rho(5)^4}{4}\right) = c\rho\pi(5)^4.$$

7 pts. to here

Therefore $$U = \frac{c\rho\pi(5)^4}{c\rho\frac{4}{3}\pi(5)^3} = \boxed{\frac{15}{4}}.$$

8 pts. to here

Math 325

Exam I

Fall 2010

$n = 37$

$\mu = 66.4$

$\sigma = 17.3$

Distribution of Scores:	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87 – 100	A	A	3
73 – 86	B	B	12
60 – 72	C	B	9
50 – 59	C	C	7
0 – 49	F	D	6