Mathematics 325 | Exam I Retest Spring 2011 | Name: Dr. Grow (1 pt.)

1.(33 pts.) (a) Verify that $u(x,y)=e^{x-y}$ is a particular solution (i.e. one involving no arbitrary functions) of the nonhomogeneous partial differential equation $yu_x + xu_y = (y-x)e^{x-y}$.

(b) Find the general solution of the homogeneous partial differential equation $yu_x + xu_y = 0$.

(c) Find the solution of $yu_x + xu_y = (y-x)e^{x-y}$ that satisfies the auxiliary condition $u(x,0)=x^4+e^x$ for all real x.

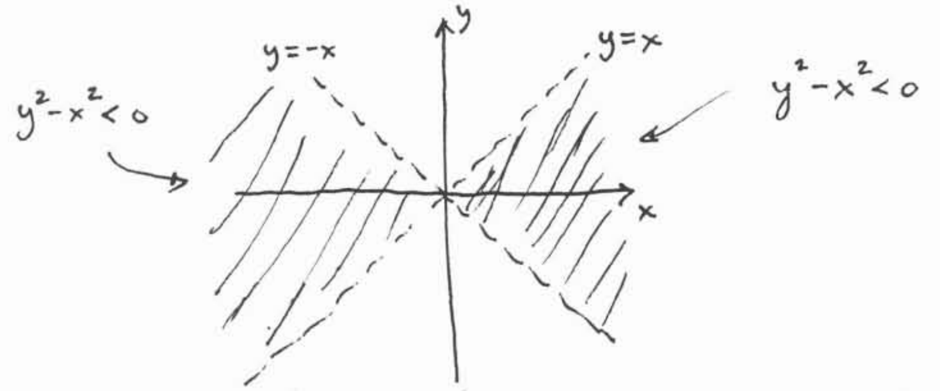
(d) What is the largest region of the xy-plane in which the solution in part (c) is uniquely determined?

(a) If $u(x,y)=e^{x-y}$ then $u_x = e^{x-y}$ and $u_y = -e^{x-y}$ so $yu_x + xu_y = ye^{x-y} - xe^{x-y} \overset{\checkmark}{=} (y-x)e^{x}$

(b) The solution to $a(x,y)u_x + b(x,y)u_y = 0$ is constant along the characteristic curves satisfying $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$. For the PDE in (b) this is $\frac{dy}{dx} = \frac{x}{y}$. Separating variables and integrating yields $\frac{y^2}{2} = \int y\,dy = \int x\,dx = \frac{x^2}{2} + c_1$ or equivalently $y^2 - x^2 = c$. Along such a characteristic curve the solution u is constant: $u(x,y) = u(x, \pm\sqrt{c+x^2}) = u(0, \pm\sqrt{c}) = f(c)$, where f is an arbitrary C^1-function of a single real variable. Thus $\boxed{u(x,y) = f(y^2-x^2)}$ is the general solution of the (homogeneous linear) PDE in (b).

(c) The general solution of the nonhomogeneous linear PDE in (a) is $u(x,y) = u_p(x,y) + u_c(x,y)$ where u_p is any particular solution of the nonhomogeneous PDE in (a) and u_c is the general solution of the associated homogeneous PDE in (b). Thus $u(x,y) = e^{x-y} + f(y^2-x^2)$ is the general solution of the nonhomogeneous PDE in (a). We use the auxiliary condition to determine f.
$x^4 + e^x = u(x,0) = e^{x-0} + f(0^2 - x^2) = e^x + f(-x^2)$ for all real x. Therefore $f(z) = z^2$ for all $z \le 0$. Consequently $\boxed{u(x,y) = e^{x-y} + (y^2-x^2)^2}$ solves the problem in (c).

(d) Since the function f is determined only for nonpositive arguments, the solution in (c) is uniquely determined only when $\boxed{y^2 - x^2 \le 0}$.



2.(33 pts.) (a) Classify the second order linear partial differential equation $u_{xx} - 4u_{xt} + 4u_{tt} = 0$ as elliptic, parabolic, or hyperbolic.

(b) Find the general solution of $u_{xx} - 4u_{xt} + 4u_{tt} = 0$ in the xt-plane.

(c) Find the solution of $u_{xx} - 4u_{xt} + 4u_{tt} = 0$ in the xt-plane satisfying $u(x,0) = xe^{2x} + 4x^2$ and $u_t(x,0) = (x-2)e^{2x} + 4x$ for all real x.

(a) $B^2 - 4AC = (-4)^2 - 4(1)(4) = 0$ so the PDE is $\boxed{\text{parabolic}}$.

(b) The PDE can be written as $\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)u = 0$. This suggests the change of coordinates: $\xi = \beta x - \alpha t = -2x - t$ and $\eta = \alpha x + \beta t = x - 2t$, or equivalently $\xi = 2x + t$, $\eta = x - 2t$. Then by the chain rule, $\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta}$. Therefore $\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t} = \cancel{2\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta} - 2\left(\cancel{\frac{\partial}{\partial \xi}} - 2\frac{\partial}{\partial \eta}\right) = 5\frac{\partial}{\partial \eta}$, so the PDE is equivalent to $\left(5\frac{\partial}{\partial \eta}\right)\left(5\frac{\partial}{\partial \eta}\right)u = 0$ or $\frac{\partial^2 u}{\partial \eta^2} = 0$. Integrating once gives $\frac{\partial u}{\partial \eta} = c_1(\xi)$ and integrating again, $u = \int c_1(\xi)\,d\eta = \eta c_1(\xi) + c_2(\xi)$. That is,

$\boxed{u(x,t) = (x-2t)f(2x+t) + g(2x+t)}$ is the general solution of the PDE in (b); here f and g are arbitrary C^2-functions of a single real variable.

(c) We use the two auxiliary conditions to determine the two arbitrary functions f and g.

① $xe^{2x} + 4x^2 = u(x,0) = (x - 2(0))f(2x+0) + g(2x+0) = xf(2x) + g(2x)$ for all real x.

$u_t(x,t) = -2f(2x+t) + (x-2t)f'(2x+t) + g'(2x+t)$

② $(x-2)e^{2x} + 4x = u_t(x,0) = -2f(2x+0) + (x-2(0))f'(2x+0) + g'(2x+0) = -2f(2x) + xf'(2x) + g'(2x)$

Differentiating ① gives:

①′ $2xe^{2x} + e^{2x} + 8x = f(2x) + 2xf'(2x) + 2g'(2x)$ for all real x.

Multiplying ② by -2 and adding to ①′ gives:

$$\cancel{-2xe^{2x}} + 4e^{2x} - \cancel{8x} + \cancel{2xe^{2x}} + e^{2x} + \cancel{8x} = 4f(2x) - \cancel{2xf'(2x)} - \cancel{2g'(2x)} + f(2x) + \cancel{2xf'(2x)} + \cancel{2g'(2x)}$$

or $5e^{2x} = 5f(2x)$ so $f(z) = e^z$ for all real z.

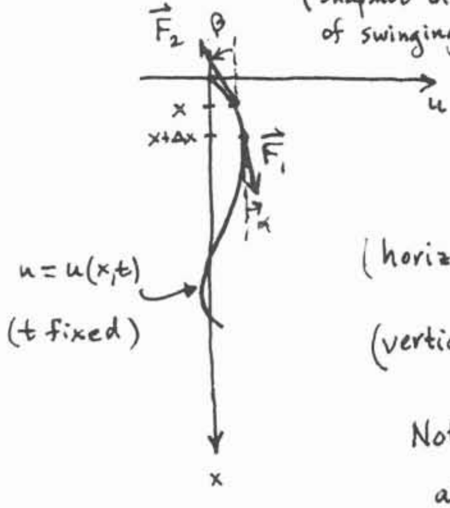
Substituting in ① produces $xe^{2x} + 4x^2 = xf(2x) + g(2x) = xe^{2x} + g(2x)$ so $g(z) = z^2$ for all real z. Therefore

$$\boxed{u(x,t) = (x-2t)e^{2x+t} + (2x+t)^2}$$

solves the initial value problem in (c).

3.(33 pts.) A flexible chain of length L is hanging from one end $x = 0$ but oscillates horizontally. Let the x – axis point downward and the u – axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Derive the partial differential equation satisfied by the chain assuming the oscillations are small.

(Snapshot in time of swinging chain.)



We apply Newton's second law, $\vec{F}_{net} = m\vec{a}$, to the segment of chain between x and $x+\Delta x$.

(horizontal component) ① $\quad |\vec{F}_1|\sin(\alpha) - |\vec{F}_2|\sin(\beta) = \int_x^{x+\Delta x} \rho u_{tt}(s,t)\,ds$

(vertical component) ② $\quad |\vec{F}_1|\cos(\alpha) - |\vec{F}_2|\cos(\beta) = 0$

Notes: ② is a very poor assumption and we will avoid its use if at all possible. The linear density of the chain is assumed constant, say ρ.

From analytic geometry $\tan(\alpha) = u_x(x+\Delta x,t)$ so $\sec(\alpha) = \sqrt{1+\tan^2\alpha} = \sqrt{1+u_x^2(x+\Delta x,t)}$ and hence $\sin(\alpha) = \dfrac{\tan(\alpha)}{\sec(\alpha)} = \dfrac{u_x(x+\Delta x,t)}{\sqrt{1+u_x^2(x+\Delta x,t)}}$. Similarly $\sin(\beta) = \dfrac{u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}$.

By assumption, $|\vec{F}_1| = (L-(x+\Delta x))\rho g$ and $|\vec{F}_2| = (L-x)\rho g$ where g is the (constant) acceleration of gravity. Substituting in ① yields:

$$\frac{(L-(x+\Delta x))\rho g\, u_x(x+\Delta x,t)}{\sqrt{1+u_x^2(x+\Delta x,t)}} - \frac{(L-x)\rho g\, u_x(x,t)}{\sqrt{1+u_x^2(x,t)}} = \int_x^{x+\Delta x} \rho u_{tt}(s,t)\,ds .$$

Dividing both sides of the previous equation by Δx and letting $\Delta x \to 0$ leads to:

$$\frac{\partial}{\partial x}\left(\frac{(L-x)u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}\right) = \frac{1}{g}u_{tt}(x,t) .$$

If the oscillations are small then $1+u_x^2(x,t) \approx 1$ so the PDE becomes

$$\boxed{\frac{\partial}{\partial x}\Big((L-x)\,u_x(x,t)\Big) = \frac{1}{g}u_{tt}(x,t)} .$$