Mathematics 325 Exam I Spring 2011 Name: Dr. Grow (1 pt.)

1.(33 pts.) (a) Verify that $u(x,y)=e^{2x+y}$ is a particular solution (i.e. one involving no arbitrary functions) of the nonhomogeneous partial differential equation $xyu_x+(1-y^2)u_y=(1+2xy-y^2)e^{2x+y}$.

(b) Find the general solution of the homogeneous partial differential equation $xyu_x+(1-y^2)u_y=0$.

(c) Find the solution of $xyu_x+(1-y^2)u_y=(1+2xy-y^2)e^{2x+y}$ that satisfies the auxiliary condition $u(x,0)=e^{2x}+e^{-x^2}$ for all real x.

(d) What is the largest region of the xy – plane in which the solution in part (c) is uniquely determined?

4 pts. (a) If $u(x,y)=e^{2x+y}$ then $u_x=2e^{2x+y}$ and $u_y=e^{2x+y}$ so

$$xyu_x+(1-y^2)u_y = 2xye^{2x+y}+(1-y^2)e^{2x+y} \overset{\checkmark}{=} (1+2xy-y^2)e^{2x+y}.$$

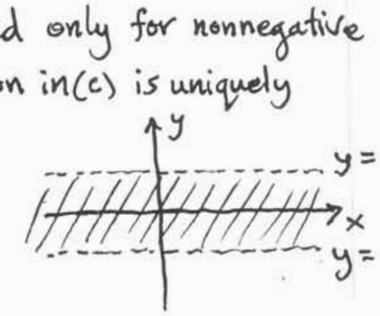
13 pts. (b) Along a characteristic curve $\frac{dy}{dx}=\frac{b(x,y)}{a(x,y)}$, solutions to $a(x,y)u_x+b(x,y)u_y=0$ are constant. For the pde in (b) the characteristic curves are given by $\frac{dy}{dx}=\frac{1-y^2}{xy}$. Separating variables yields $\frac{y\,dy}{1-y^2}=\frac{dx}{x}$. Integrating both sides produces $-\frac{1}{2}\ln|1-y^2| = \ln|x|+c_1$ and simplifying gives $x^2(1-y^2)=C$ where C is an abitrary positive constant $(C=e^{-2c_1})$. Along such a characteristic curve, a solution u is constant: $u(x,y) = u\left(\pm\sqrt{\frac{C}{1-y^2}},y\right)=u\left(\pm\sqrt{C},0\right)=f(C)$ where f is an arbitrary C^1-function of a single real variable. Thus the pde in (b) has general solution $\boxed{u(x,y)=f\left(x^2(1-y^2)\right)}$.

13 pts. (c) The general solution to the nonhomogeneous (linear) pde in (c) has the form $u(x,y)=u_p(x,y)+u_c(x,y)$ where u_p is a particular solution of the nonhomogeneous equation and u_c is the general solution of the associated homogeneous equation $xyu_x+(1-y^2)u_y=0$. By parts (a) and (b), $u(x,y)=e^{2x+y}+f(x^2(1-y^2))$. We apply the auxiliary condition to find f:

$$e^{2x}+e^{-x^2}=u(x,0)=e^{2x+0}+f\left(x^2(1-0^2)\right)=e^{2x}+f(x^2) \quad \text{for all } -\infty<x<\infty.$$

Consequently $f(z)=e^{-z}$ for all $z\geq 0$ and hence $\boxed{u(x,y)=e^{2x+y}+e^{-x^2(1-y^2)}}$ solves (c).

3 pts. (d) Note that the function f in part (c) is uniquely determined only for nonnegative arguments; i.e. $f(z)=e^{-z}$ for $z\geq 0$. Therefore the solution in (c) is uniquely determined if $x^2(1-y^2)\geq 0$, or equivalently $\boxed{-1\leq y\leq 1.}$



2.(33 pts.) (a) Classify the second order linear partial differential equation $u_{tt} - c^2 u_{xx} = 0$ as elliptic, parabolic, or hyperbolic.

(b) Find the general solution of $u_{tt} - c^2 u_{xx} = 0$ in the xt-plane.

(c) Find the solution of $u_{tt} - c^2 u_{xx} = 0$ in the xt-plane satisfying $u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$ for all real x. (Here φ and ψ are given C^2 and C^1 functions, respectively, of a single real variable.)

4 pts. (a) $B^2 - 4AC = 0^2 - 4(-c^2)(1) > 0$. The pde is $\boxed{\text{hyperbolic}}$.

14 pts. (b) The pde is equivalent to $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$. Let $\xi = +ct + x$, $\eta = ct - x$. Then $\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}$ so the pde is transformed into the equivalent pde

$$0 = \left[c\left(\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right) - c\left(\cancel{\frac{\partial}{\partial \xi}} - \frac{\partial}{\partial \eta}\right)\right]\left[c\left(\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}}\right) + c\left(\frac{\partial}{\partial \xi} - \cancel{\frac{\partial}{\partial \eta}}\right)\right]u = 4c^2\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right).$$

Integration yields $\frac{\partial u}{\partial \xi} = h(\xi)$ and another integration gives $u = \int h(\xi)\,d\xi + c(\eta)$. That is, $u = f(\xi) + \tilde{g}(\eta)$ where f and $\tilde{g}$ are C^2 functions of a single real variable. In terms of the original variables x and t, the general solution is $\boxed{u(x,t) = f(x+ct) + g(x-ct)}$

1[illegible] pts. (c) The initial condition $u(x,0) = \varphi(x)$ leads to $f(x) + g(x) \overset{①}{=} \varphi(x)$ for all $-\infty < x < \infty$. Note that $u_t(x,t) = cf'(x+ct) - cg'(x-ct)$ so the initial condition $u_t(x,0) = \psi(x)$ leads to $cf'(x) - cg'(x) \overset{②}{=} \psi(x)$ for all $-\infty < x < \infty$. Differentiating ① produces $f'(x) + g'(x) \overset{③}{=} \varphi'(x)$ forall $-\infty < x < \infty$. Multiplying ③ by c and adding to ② yields $2cf'(x) = c\varphi'(x) + \psi(x)$. Dividing by $2c$ and integrating, we have $f(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds + c_1$ where c_1 is an arbitrary constant. From this and ① it follows that $g(x) = \varphi(x) - f(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds - c_1$. Therefore

$$u(x,t) = f(x+ct) + g(x-ct) = \frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi(s)\,ds + \frac{1}{2}\varphi(x-ct) - \frac{1}{2c}\int_0^{x-ct}\psi(s)\,ds.$$

Simplifying yields d'Alembert's formula:

$$\boxed{u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds.}$$

3.(33 pts.) In linearized gas dynamics, the velocity $\mathbf{v}$ of the disturbance and the density ρ of the air satisfy $\frac{\partial \mathbf{v}}{\partial t}+\frac{c_0^2}{\rho_0}\operatorname{grad}(\rho)\overset{①}{=}\mathbf{0}$ and $\frac{\partial \rho}{\partial t}+\rho_0 \operatorname{div}(\mathbf{v})\overset{②}{=}0$, where ρ_0 is the density of still air and c_0 is the speed of sound in still air. Suppose that $\operatorname{curl}(\mathbf{v})=\mathbf{0}$ at time $t=0$.

(a) Show that $\operatorname{curl}(\mathbf{v})=\mathbf{0}$ at all later times.

(b) Show that ρ and the components of $\mathbf{v}$ satisfy the three-dimensional wave equation.

Recall the definitions of the curl and divergence of a vector field $\vec{v}$:

$$\operatorname{curl}(\vec{v}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \overset{③}{=} \hat{\imath}\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)+\hat{\jmath}\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)+\hat{k}\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right) \quad (2)$$

and $$\operatorname{div}(\vec{v}) = \frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}. \quad (2)$$

15 pts. (a) Fix a point (x,y,z) in $\mathbb{R}^3$ and $t>0$. The Fundamental Theorem of Calculus and equation ① imply

$$\vec{v}(x,y,z,t)-\vec{v}(x,y,z,0) = \int_0^t \frac{\partial \vec{v}}{\partial t}(x,y,z,\tau)\,d\tau = -\frac{c_0^2}{\rho_0}\int_0^t \operatorname{grad}(\rho(x,y,z,\tau))\,d\tau.$$

Therefore

$$\operatorname{curl}\left(\vec{v}(x,y,z,t)\right) = \operatorname{curl}\left(\vec{v}(x,y,z,0) - \frac{c_0^2}{\rho_0}\int_0^t \operatorname{grad}(\rho(x,y,z,\tau))\,d\tau\right)$$

$$\overset{④}{=} \operatorname{curl}\left(\vec{v}(x,y,z,0)\right) - \frac{c_0^2}{\rho_0}\int_0^t \operatorname{curl}\left(\operatorname{grad}(\rho(x,y,z,\tau))\right)d\tau.$$

But

$$\operatorname{curl}(\operatorname{grad}(\rho)) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z} \end{vmatrix} = \hat{\imath}\left(\frac{\partial^2 \rho}{\partial y \partial z}-\frac{\partial^2 \rho}{\partial z \partial y}\right)+\hat{\jmath}\left(\frac{\partial^2 \rho}{\partial z \partial x}-\frac{\partial^2 \rho}{\partial x \partial z}\right)+\hat{k}\left(\frac{\partial^2 \rho}{\partial x \partial y}-\frac{\partial^2 \rho}{\partial y \partial x}\right)=\vec{0}$$

and $\operatorname{curl}(\vec{v}(x,y,z,0))=\vec{0}$ so substituting in ④ yields $\operatorname{curl}(\vec{v}(x,y,z,t))=\vec{0}$ for $t>0$.

18 pts. (b) Using equations ② and ① yields

(cont.)

$$\frac{\partial^2\rho}{\partial t^2} \overset{(2)}{=} \frac{\partial}{\partial t}\left(-\rho_0 \operatorname{div}(\vec{v})\right) = -\rho_0 \frac{\partial}{\partial t}\left(\operatorname{div}(\vec{v})\right) = -\rho_0 \operatorname{div}\left(\frac{\partial \vec{v}}{\partial t}\right) \overset{(1)}{=} -\rho_0 \operatorname{div}\left(-\frac{c_0^2}{\rho_0}\operatorname{grad}(\rho)\right)$$

$$= c_0^2 \operatorname{div}\left(\operatorname{grad}(\rho)\right) = c_0^2\left(\frac{\partial}{\partial x}\left(\frac{\partial\rho}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\rho}{\partial z}\right)\right).$$ That is,

$\rho_{tt} - c_0^2\nabla^2\rho = 0$ so ρ satisfies the 3-D wave equation.

Similarly equations (1) and (2) imply

$$\frac{\partial^2\vec{v}}{\partial t^2} \overset{(1)}{=} \frac{\partial}{\partial t}\left(-\frac{c_0^2}{\rho_0}\operatorname{grad}(\rho)\right) = -\frac{c_0^2}{\rho_0}\operatorname{grad}\left(\frac{\partial\rho}{\partial t}\right) \overset{(2)}{=} -\frac{c_0^2}{\rho_0}\operatorname{grad}\left(-\rho_0 \operatorname{div}(\vec{v})\right) = c_0^2 \operatorname{grad}(\operatorname{div}(\vec{v})).$$

Writing out the first component of the previous vector pde yields

$$\frac{\partial^2 v_1}{\partial t^2} \overset{(4)}{=} c_0^2\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) = c_0^2\left[\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_2}{\partial x\partial y} + \frac{\partial^2 v_3}{\partial x\partial z}\right].$$

But $\operatorname{curl}(\vec{v}) = \vec{0}$ for all $t \geq 0$ by part (a), so it follows from (3) that

$$\frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) = 0 = \frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right).$$

Rearranging yields

$$\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \overset{(5)}{=} \frac{\partial^2 v_2}{\partial y\partial x} + \frac{\partial^2 v_3}{\partial z\partial x}.$$

Substituting from (5) into (4) leads to

$$\frac{\partial^2 v_1}{\partial t^2} = c_0^2\left[\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right],$$

or in other words, v_1 satisfies the 3-D wave equation $v_{tt} - c_0^2\nabla^2 v = 0$.

The arguments showing that v_2 and v_3 satisfy the 3-D wave equation are completely analogous.

Math 325

Exam I

Spring 2011

mean: 50.6

standard deviation: 18.2

n: 25

<u>Distribution of Scores</u>	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87 – 100	A	A	1
73 – 86	B	B	3
60 – 72	C	B	2
50 – 59	C	C	4
0 – 49	F	D	15

A's	1
B's	4
C's	5
D's	9
F's	6