Mathematics 325 Exam I Name: Dr. Grow

1.(25 pts.) Find all second-degree polynomial functions of two real variables,

$$u(x,t) = ax^2 + bxt + ct^2 + dx + et + f$$

where a, b, c, d, e, and f are real constants, which are solutions in the xt – plane of the one-dimensional diffusion equation

$$u_t - ku_{xx} = 0.$$

$$\left.\begin{aligned} u_t &= bx + 2ct + e \\ u_x &= 2ax + bt + d \\ u_{xx} &= 2a \end{aligned}\right\}$$

6 pts. to here.

We want $u_t = ku_{xx}$ so substituting gives

$$bx + 2ct + e = k2a.$$

12 pts. to here.

This is to hold for all $-\infty < x < \infty$ and $-\infty < t < \infty$.

Therefore we must have "like" coefficients equal on the left and right sides of this identity. That is, $b = 0$, $2c = 0$, and $e = 2ka$. Thus

18pts. to here.

$$u(x,t) = ax^2 + dx + 2kat + f$$

$$\boxed{u(x,t) = a(x^2 + 2kt) + dx + f}$$

25 pts. to here.

where a, d, and f are arbitrary real constants.

2.(25 pts.) Find the general solution of

$$yu_x - xu_y = 0$$

in the xy – plane. Sketch several characteristic curves of this partial differential equation.

The solution $u = u(x,y)$ is constant along all curves in the xy-plane which satisfy the characteristic equation of this pde: $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} = \frac{-x}{y}$. 6 pts. to here.

Separating variables and integrating yields

$$\frac{y^2}{2} = \int y\,dy = \int -x\,dx = -\frac{x^2}{2} + c_1$$

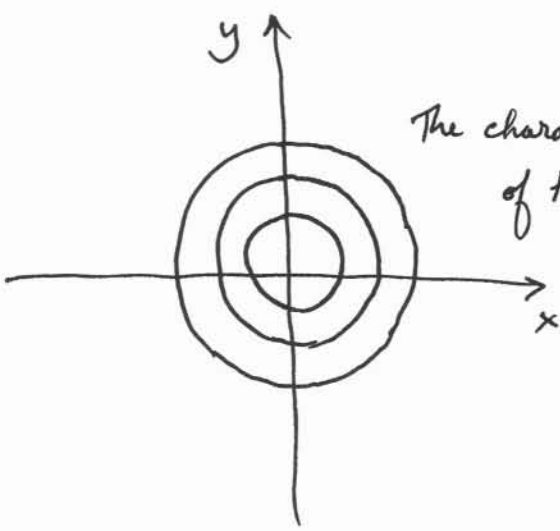
$$\Rightarrow \quad x^2 + y^2 = c. \qquad (\text{where } c = 2c_1)$$

12 pts. to here.

Along each such curve we have

$$u(x,y) = u(x, \pm\sqrt{c - x^2}) = u(0, \pm\sqrt{c}) = f(c).$$

Therefore $\boxed{u(x,y) = f(x^2+y^2)}$ where f is an arbitrary differentiable function of a single real variable. 20 pts. to here.



The characteristic curves $x^2+y^2 = c$ of the pde are circles centered at the origin.

25pts. to here.

3.(25 pts.) Consider the linearized gas dynamics equations

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{c_0^2}{\rho_0}\operatorname{grad}(\rho)=\mathbf{0}$$

$$\frac{\partial \rho}{\partial t}+\rho_0 \operatorname{div}(\mathbf{v})=0$$

where ρ_0 is the density and c_0 is the speed of sound in still air. Verify that if $\operatorname{curl}(\mathbf{v})=\mathbf{0}$ when $t=0$, then $\operatorname{curl}(\mathbf{v})=\mathbf{0}$ at all later times.

Suppose $\vec{v}=\vec{v}(x,y,z,t)$ and $\rho=\rho(t)$ are solutions to the gas dynamics equation. Consider the function

$$\vec{w}=\operatorname{curl}(\vec{v}(x,y,z,t)).$$

Then $\dfrac{\partial \vec{w}}{\partial t}=\dfrac{\partial}{\partial t}\operatorname{curl}(\vec{v}(x,y,z,t))=\operatorname{curl}\left(\dfrac{\partial \vec{v}}{\partial t}\right)$ because $\vec{v}=\vec{v}(x,y,z,t)$ is a C^2-function so $\dfrac{\partial}{\partial t}\left(\dfrac{\partial v_3}{\partial y}-\dfrac{\partial v_2}{\partial z}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial v_3}{\partial t}\right)-\dfrac{\partial}{\partial z}\left(\dfrac{\partial v_2}{\partial t}\right)$, etc. But then

$$\frac{\partial \vec{w}}{\partial t}=\operatorname{curl}\left(-\frac{c_0^2}{\rho_0}\operatorname{grad}(\rho)\right)=-\frac{c_0^2}{\rho_0}\operatorname{curl}(\operatorname{grad}(\rho))=\vec{0}$$

from the first gas dynamic equation and the fact that the curl of a gradient of a C^2-function is zero. It follows that the function $\vec{w}$ is independent of t; i.e. for all $t\geq 0$ and all $(x,y,z)\in\mathbb{R}^3$,

$$\operatorname{curl}(\vec{v}(x,y,z,t))=\vec{w}(x,y,z,t)=\vec{w}(x,y,z,0)=\operatorname{curl}(\vec{v}(x,y,z,0))=\vec{0}.$$

4.(25 pts.) Consider the partial differential equation

(*) $$u_{xx} - 3u_{xt} - 4u_{tt} = 0.$$

(a) Classify (*) as elliptic, hyperbolic, or parabolic.

(b) Find the general solution of (*) in the xt-plane.

(c) If ϕ and ψ are C^2 and C^1 functions of a single real variable, respectively, find the solution of (*) which satisfies the initial conditions

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for all } -\infty < x < \infty.$$

5 pts. (a) $B^2 - 4AC = (-3)^2 - 4(1)(-4) = 9 + 16 = 25 > 0$. $\boxed{(*) \text{ is hyperbolic}}$.

(b) (*) is equivalent to $\left(\frac{\partial^2}{\partial x^2} - 3\frac{\partial^2}{\partial x \partial t} - 4\frac{\partial^2}{\partial t^2}\right)u = 0$ and thus to

$\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u = 0$. The change-of-coordinates $\begin{cases} \xi = -(\beta x - \alpha t) = -(-4x - t) \\ \eta = \delta x - \gamma t = x - t \end{cases}$

10 pts. to here.

transforms (*) into $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$ so the general solution of (*) is

15 pts. to here. $\boxed{u = f(\xi) + g(\eta) = f(4x+t) + g(x-t)}$ where f and g are arbitrary twice-differentiable functions of a single real variable.

(c) Using (b) and the initial conditions in (c), we have

(1) $\varphi(x) = u(x,0) = f(4x) + g(x) \qquad (-\infty < x < \infty)$

(2) $\psi(x) = u_t(x,0) = f'(4x) - g'(x) \qquad (-\infty < x < \infty)$

Differentiating (1) yields

(1′) $\varphi'(x) = 4f'(4x) + g'(x) \qquad (-\infty < x < \infty).$

Adding (2) and (1′) produces

(3) $\varphi'(x) + \psi(x) = 5f'(4x) \qquad (-\infty < x < \infty)$

or equivalently

(4) $f'(z) = \frac{1}{5}\varphi'\left(\frac{z}{4}\right) + \frac{1}{5}\psi\left(\frac{z}{4}\right) \qquad (-\infty < z < \infty).$

Integrating yields (OVER)

$$(5) \quad f(z) = \frac{4}{5}\varphi\left(\frac{z}{4}\right) + \frac{4}{5}\int_0^{z/4} \psi(\xi)\,d\xi + c_1 \qquad (-\infty < z < \infty).$$

Therefore, using (5) and (1) we have

$$(6) \quad g(x) = \varphi(x) - f(4x) = \varphi(x) - \left(\frac{4}{5}\varphi(x) + \frac{4}{5}\int_0^{x} \psi(\xi)\,d\xi + c_1\right)$$

$$= \frac{1}{5}\varphi(x) - \frac{4}{5}\int_0^{x} \psi(\xi)\,d\xi - c_1 \qquad (-\infty < x < \infty)$$

It follows from (b), (5), and (6) that

$$u(x,t) = f(4x+t) + g(x-t)$$

$$= \frac{4}{5}\varphi\left(\frac{4x+t}{4}\right) + \frac{4}{5}\int_0^{(4x+t)/4} \psi(\xi)\,d\xi + \cancel{c_1} + \frac{1}{5}\varphi(x-t) - \frac{4}{5}\int_0^{x-t} \psi(\xi)\,d\xi - \cancel{c_1}$$

25 pts. to here.

$$\boxed{u(x,t) = \frac{1}{5}\left[4\varphi\left(x+\tfrac{1}{4}t\right) + \varphi(x-t)\right] + \frac{4}{5}\int_{x-t}^{x+\frac{1}{4}t} \psi(\xi)\,d\xi}\,.$$

Bonus.(25 pts.) A homogeneous solid material occupying $D=\{(x,y,z)\in\mathbb{R}^3:\ 4\le x^2+y^2+z^2\le 100\}$ is completely insulated and its initial temperature at position (x,y,z) in D is $200/\sqrt{x^2+y^2+z^2}$.

(a) Write (without proof or derivation) the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) in D and time $t\ge 0$.

(b) Use Gauss' divergence theorem to help show that the heat energy $H(t)=\iiint_D c\rho u(x,y,z,t)\,dV$ of the material in D at time t is a constant function of time. Here c and ρ denote the (constant) specific heat and mass density, respectively, of the material in D.

(c) Compute the (constant) steady-state temperature that the material in D reaches after a long time.

(a)

9 pts.

$$\begin{cases} u_t - k(\overbrace{u_{xx}+u_{yy}+u_{zz}}^{\nabla^2 u}) \overset{①}{=} 0 & \text{if } 4 < x^2+y^2+z^2 < 100 \text{ and } t>0 \\ \nabla u\cdot\vec{n} \overset{②}{=} 0 & \text{if } x^2+y^2+z^2=4 \text{ or } x^2+y^2+z^2=100 \text{ and } t\ge 0 \\ u(x,y,z,0) \overset{③}{=} \dfrac{200}{\sqrt{x^2+y^2+z^2}} & \text{if } 4\le x^2+y^2+z^2\le 100 \end{cases}$$

(b)

8 pts.

$$H'(t)=\frac{d}{dt}\iiint_D c\rho u(x,y,z,t)\,dV = \iiint_D \frac{\partial}{\partial t}(c\rho u)\,dV = c\rho\iiint_D u_t\,dV$$

$$\overset{①}{=} kc\rho\iiint_D \nabla^2 u\,dV = kc\rho\iiint_D \nabla\cdot(\nabla u)\,dV \overset{\text{Gauss' Divergence Thm.}}{=} kc\rho\iint_{\partial D}\nabla u\cdot\vec{n}\,dV \overset{②}{=} 0.$$

Therefore the heat energy $H=H(t)$ is a constant function of t.

(c) Let $U=\lim_{t\to\infty}u(x,y,z,t)$ be the constant steady-state temperature

8 pts.

that the material in D reaches after a long time. From part (b), $H(0)=H(t)$ for all $t\ge 0$. Therefore

$$H(0)=\lim_{t\to\infty}H(t)=\lim_{t\to\infty}\iiint_D c\rho u(x,y,z,t)\,dV = c\rho\iiint_D \lim_{t\to\infty}u(x,y,z,t)\,dV$$

$$= c\rho\iiint_D U\,dV = c\rho U\,\mathrm{vol}(D).$$

(OVER)

Solving for $\overline{U}$ we find

$$\overline{U} = \frac{H(0)}{c\rho \, \mathrm{vol}(D)}.$$

But computations yield

$$\mathrm{vol}(D) = \frac{4}{3}\pi r_1^3 - \frac{4}{3}\pi r_2^3 = \frac{4}{3}\pi\left(10^3 - 2^3\right) = \frac{3968}{3}\pi$$

and

$$H(0) = \iiint_D c\rho u(x,y,z,0)\,dV = c\rho \iiint_D \frac{200}{\sqrt{x^2+y^2+z^2}}\,dV$$

$$= c\rho \int_0^{2\pi}\int_0^{\pi}\int_2^{10} \frac{200}{r} r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

$$= c\rho(2\pi)(-\cos\varphi)\Big|_0^{\pi}\left(100r^2\right)\Big|_2^{10}$$

$$= 38{,}400\,\pi c\rho.$$

Therefore

$$\boxed{\overline{U} = \frac{38{,}400\pi c\rho}{c\rho\left(\frac{3968}{3}\pi\right)} = \frac{900}{31} \doteq 29.0}.$$

Math 325

Exam I

Summer 2009

n : 15

standard deviation: 20.7

mean: 72.6

Distribution of Scores:

		frequency
87 – 100	A	3
73 – 86	B	8
60 – 72	C (graduate), B (undergraduate)	1
50 – 59	C	1
0 – 49	F (graduate), D (undergraduate)	2