Mathematics 325 | Exam I | Name: Dr. Grow

Summer 2010

1.(25 pts.) (a) Find the characteristic curves of $yu_x - xu_y = 0$ in the xy – plane.

(b) Sketch and identify two characteristic curves of this partial differential equation.

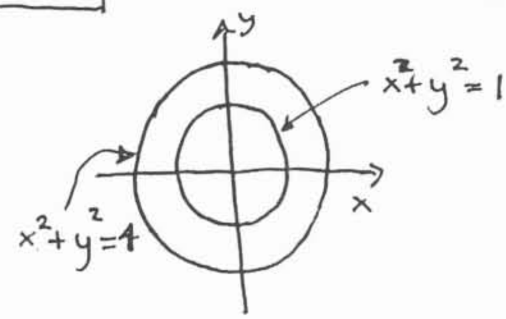
(c) Write the general solution of this partial differential equation in the xy – plane.

(d) Find the solution of this partial differential equation which satisfies $u(x,0) = x^6$ for all real x.

(e) In what region of the xy – plane is the solution in part (d) uniquely determined?

6 pts. (a) The characteristic curves of the p.d.e. satisfy $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} = \frac{-x}{y}$.
Separating variables gives $y\,dy = -x\,dx$ and integrating both sides yields
$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c_1$, or equivalently $\boxed{x^2+y^2 = c.}$ ($c = 2c_1$ is an arbitrary constant)

3 pts. (b) The characteristic curves are $\boxed{\text{circles}}$.



8 pts. (c) The solutions to the p.d.e. are constant along characteristic curves. Along such a curve $x^2+y^2=c$ we have

$$u(x,y) = u(x, \pm\sqrt{c-x^2}) = u(0, \pm\sqrt{c}) = f(c).$$

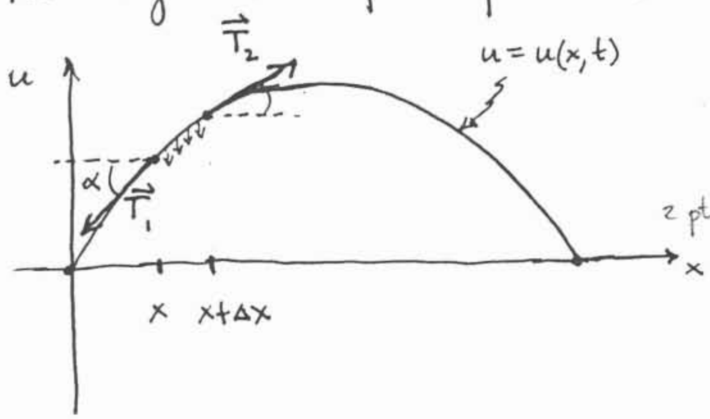
Therefore the general solution to the p.d.e. in the xy-plane is $\boxed{u(x,y) = f(x^2+y^2)}$ where f is any C^1-function of a single real variable.

6 pts. (d) $x^6 = u(x,0) = f(x^2+0^2) = f(x^2)$ for all real x. Thus $f(t) = t^3$ for every $t \geq 0$, so $\boxed{u(x,y) = f(x^2+y^2) = (x^2+y^2)^3}$ for all (x,y) in the plane.

2 pts. (e) The solution in part (d) is uniquely determined in $\boxed{\text{the entire } xy\text{-plane.}}$

2.(25 pts.) An elastic string is held fixed at the endpoints and plucked. Assuming that air exerts a resistance at each point of the string which is proportional to the string's velocity at that point, give a careful derivation of the partial differential equation that governs the small vibrations of the string.

Fix $t > 0$ and suppose the profile of the string at this instant is as indicated in the diagram. Consider the segment of string between x and $x+\Delta x$. The contact forces $\vec{T}_1$ and $\vec{T}_2$ that act at the end points of the segment are tangential to the string at the respective points. If α and β are the angles these contact forces make with respect to the horizontal, then



2 pts. ① $\tan(\alpha) =$ slope of the tangent line to the profile at x

$= u_x(x,t)$

so

4 pts. to here. $\sin(\alpha) \overset{②}{=} \dfrac{u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}$ and $\cos(\alpha) \overset{③}{=} \dfrac{1}{\sqrt{1+u_x^2(x,t)}}$.

Similarly,

6 pts. to here. $\sin(\beta) \overset{④}{=} \dfrac{u_x(x+\Delta x,t)}{\sqrt{1+u_x^2(x+\Delta x,t)}}$ and $\cos(\beta) \overset{⑤}{=} \dfrac{1}{\sqrt{1+u_x^2(x+\Delta x,t)}}$

Applying Newton's second law, $m\vec{a} = \vec{F}_{net}$, to the system of particles comprising the segment of string from x to $x+\Delta x$, we have upon resolving into components:

8 pts. to here. (Horizontal) $|\vec{T}_2|\cos(\beta) - |\vec{T}_1|\cos(\alpha) \overset{⑥}{=} 0$

15 pts. to here (2 + 3 + 2) (Vertical) $|\vec{T}_2|\sin(\beta) - |\vec{T}_1|\sin(\alpha) - \int_x^{x+\Delta x} r u_t(\xi,t)\,d\xi \overset{⑦}{=} \int_x^{x+\Delta x} \rho(\xi) u_{tt}(\xi,t)\,d\xi$

(OVER)

where r is a positive constant and $\rho(\xi)$ is the linear density of the string at ξ in the interval $[x, x+\Delta x]$. (Here we have made use of the assumption that the air exerts a resistive force at each point that is proportional to the string's velocity u_t at that point.) Equations (3), (5), and (6) imply

17 pts. to here. (8)
$$|\vec{T}_2| = |\vec{T}_1| \frac{\sqrt{1+u_x^2(x+\Delta x,t)}}{\sqrt{1+u_x^2(x,t)}}$$

so substituting from (8) into (7) and using (2) and (4) yields

19 pts. to here (9)
$$\frac{|\vec{T}_1|}{\sqrt{1+u_x^2(x,t)}}\left(u_x(x+\Delta x,t) - u_x(x,t)\right) - \int_x^{x+\Delta x} r u_t(\xi,t)\,d\xi = \int_x^{x+\Delta x} \rho(\xi) u_{tt}(\xi,t)\,d\xi .$$

Dividing (9) by Δx and letting $\Delta x \to 0$ produces

2 ts. to here. (10)
$$\frac{|\vec{T}(x, u(x,t), u_x(x,t))|}{\sqrt{1+u_x^2(x,t)}} u_{xx}(x,t) - r u_t(x,t) = \rho(x) u_{tt}(x,t) .$$

For small vibrations, $u(x,t)$ and $u_x(x,t)$ are quite small so $\sqrt{1+u_x^2(x,t)} \approx 1$ for all $0 \le x \le L$. Thus (8) implies the magnitude of the contact force is approximately constant, say $|\vec{T}(x, u(x,t), u_x(x,t))| = \text{constant} = T_0$. Thus (10) becomes

23 pts. to here. (11)
$$\frac{T_0}{\sqrt{1+u_x^2(x,t)}} u_{xx}(x,t) - r u_t(x,t) = \rho(x) u_{tt}(x,t).$$

If the string is homogeneous, say $\rho(x) = \text{constant} = \rho_0$, and we use the small vibrations approximation, $\sqrt{1+u_x^2(x,t)} \approx 1$ for all $0 \le x \le L$, then (11) becomes

25 pts. to here. (12)
$$\boxed{T_0 u_{xx}(x,t) - r u_t(x,t) = \rho_0 u_{tt}(x,t)}$$

where T_0, r, and ρ_0 are positive constants.

3.(25 pts.) Classify the following second order partial differential equations as hyperbolic, parabolic, elliptic, or nonlinear. Find the general C^2 – solution in the plane whenever possible.

(a) $9u_{xx}+6u_{xy}+u_{yy}+100u=0$

(b) $u_{xx}+u_{yy}+uu_x=0$

(c) $u_{xx}-u_{xy}+3u_{yy}-3u_{yx}+2u_x-2u_y=0$ $\Leftrightarrow$ $u_{xx}-4u_{xy}+3u_{yy}+2u_x-2u_y=0$.

2 pts. (a) $B^2-4AC=6^2-4(9)(1)=0$ so the p.d.e. is $\boxed{\text{parabolic}}$.

2 pts. (b) The p.d.e. is $\boxed{\text{nonlinear}}$ because of the term uu_x.

2 pts. (c) $B^2-4AC=(-4)^2-4(1)(3)=4>0$ so the p.d.e. is $\boxed{\text{hyperbolic}}$.

8 pts to here. To solve the p.d.e. in (a) note that it is equivalent to $\left(3\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(3\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)u+100u=0$

10 pts. to here. Let $\begin{cases}\xi=3x+y\\ \eta=x-3y\end{cases}$. Then by the chain rule, as operators we have

$$\frac{\partial}{\partial x}=\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}=3\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\quad\text{and}\quad\frac{\partial}{\partial y}=\frac{\partial\xi}{\partial y}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta}=\frac{\partial}{\partial\xi}-3\frac{\partial}{\partial\eta}.$$

Therefore $3\frac{\partial}{\partial x}+\frac{\partial}{\partial y}=3\left(3\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)+\left(\frac{\partial}{\partial\xi}-3\frac{\partial}{\partial\eta}\right)=10\frac{\partial}{\partial\xi}$. Thus the p.d.e.

12 pts. to here. in (a) is equivalent to $\left(10\frac{\partial}{\partial\xi}\right)\left(10\frac{\partial}{\partial\xi}\right)u+100u=0\Rightarrow\frac{\partial^2u}{\partial\xi^2}+u=0$.

Consequently, $u=c_1(\eta)\cos(\xi)+c_2(\eta)\sin(\xi)$, or equivalently,

14 pts. to here.

$$\boxed{u(x,y)=f(x-3y)\cos(3x+y)+g(x-3y)\sin(3x+y)}$$

is the general solution of (a) with f and g arbitrary C^2-functions of a single real variab

16 pts. to here. To solve the p.d.e. in (c) note that it is equivalent to $0=\left(\frac{\partial}{\partial x}-3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)u+2\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right.$

Let $\begin{cases}\xi=3x+y\\ \eta=x+y\end{cases}$. Then the chain rule implies, as operators we have

18 pts. to here.

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$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

Thus $\displaystyle \frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - 3\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = -2\frac{\partial}{\partial \eta}$

and $\displaystyle \frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = 2\frac{\partial}{\partial \xi}.$

Consequently, the p.d.e. in (c) is equivalent to

$$\left(-2\frac{\partial}{\partial \eta}\right)\left(2\frac{\partial}{\partial \xi}\right)u + 2\left(2\frac{\partial}{\partial \xi}\right)u = 0$$

20 pts. to here. $$-\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) + \frac{\partial u}{\partial \xi} = 0.$$

22 pts. to here. Letting $v = \frac{\partial u}{\partial \xi}$, this becomes $\frac{\partial v}{\partial \eta} - v = 0$. Therefore $v = c(\xi)e^{\eta}$.

That is, $\displaystyle \frac{\partial u}{\partial \xi} = c(\xi)e^{\eta}$ so $\displaystyle u = \int c(\xi)e^{\eta}d\xi = e^{\eta}\int c(\xi)d\xi = e^{\eta}\left(f(\xi) + c_2(\eta)\right)$

24 pts. to here. Consequently, $u = f(\xi)e^{\eta} + g(\eta) \qquad \left(g(\eta) = e^{\eta}c_2(\eta)\right)$,

or equivalently

25 pts to here. $$\boxed{u(x,y) = f(3x+y)e^{x+y} + g(x+y)}$$

is the general solution of (c) where f and g are arbitrary C^2-functions of a single real variable.

4.(25 pts.) Let $u = u(x,t)$ be a nonconstant C^2 – solution to

$$u_{tt} - u_{xx} + u_t = 0$$

in the upper half-plane $-\infty < x < \infty$, $0 \le t < \infty$, such that for each fixed $t \ge 0$, u_t, u_x, u_{xx}, u_{xt}, and u_{tt} are square-integrable functions of x on $(-\infty, \infty)$ and $\lim_{|x| \to \infty} u_x(x,t) u_t(x,t) = 0$.

(a) Show that $G(t) = \int_{-\infty}^{\infty} \left\{ \left[u_t(x,t) \right]^2 + \left[u_x(x,t) \right]^2 \right\} dx$ is a decreasing function for $t \ge 0$.

(b) Give a physical interpretation of the result in part (a).

(a) $$G'(t) = \frac{d}{dt} \int_{-\infty}^{\infty} \left\{ \left[u_t(x,t) \right]^2 + \left[u_x(x,t) \right]^2 \right\} dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \left\{ \left[u_t(x,t) \right]^2 + \left[u_x(x,t) \right]^2 \right\} dx$$

3 pts to here. $$= 2 \int_{-\infty}^{\infty} \left[u_t(x,t) u_{tt}(x,t) + u_x(x,t) u_{tx}(x,t) \right] dx$$

But $$\int_{-\infty}^{\infty} u_x(x,t) u_{tx}(x,t)\, dx = \lim_{\substack{M \to \infty \\ N \to \infty}} \int_N^M \overbrace{u_x(x,t)}^{U} \overbrace{u_{tx}(x,t)\, dx}^{dV}$$ (Integrate by parts

$$= \lim_{\substack{M \to \infty \\ N \to -\infty}} \left(u_x(x,t) u_t(x,t) \Big|_{x=N}^{M} - \int_N^M u_t(x,t) u_{xx}(x,t)\, dx \right)$$

8 pts. to here. $$= - \int_{-\infty}^{\infty} u_t(x,t) u_{xx}(x,t)\, dx.$$

Therefore, $$G'(t) = 2 \int_{-\infty}^{\infty} \left[u_t(x,t) u_{tt}(x,t) - u_t(x,t) u_{xx}(x,t) \right] dx$$

$$= 2 \int_{-\infty}^{\infty} u_t(x,t) \left[u_{tt}(x,t) - u_{xx}(x,t) \right] dx$$

. . pts. to here. $$= -2 \int_{-\infty}^{\infty} u_t(x,t) u_t(x,t)\, dx$$ (since u solves $u_{tt} - u_{xx} + u_t = 0$

(OVER)

so $G'(t) = -2\int_{-\infty}^{\infty} [u_t(x,t)]^2 dx \leq 0$ and hence G is a

14 pts. to here. decreasing function (in the weak sense) on $t \geq 0$.

(b) The equation $u_{tt} - u_{xx} + u_t = 0$ models the small vibrations of a homogeneous elastic string with tension $T_0 = 1$, density $\rho_0 = 1$, and

20 pts. to here. damping constant $r = 1$. (See problem #2 on this exam.) The energy at time t of a solution $u = u(x,t)$ to $u_{tt} - u_{xx} + u_t = 0$ is

22 pts. to here.

$$E(t) = \int_{-\infty}^{\infty} \left[\tfrac{1}{2}\rho_0 u_t^2(x,t) + \tfrac{1}{2} T_0 u_x^2(x,t) \right] dx = \tfrac{1}{2} G(t).$$

Since damping transfers energy from the vibrating string to the

25 pts to here. environment, one expects $E(t)$ (and consequently $G(t)$) to decrease with time t.