Mathematics 325 | Exam II Spring 2011 | Name: Dr. Grow (1 pt.)

1.(33 pts.) Use the Fourier transform method to derive a formula for the solution to the diffusion equation with variable dissipation

$$u_t - ku_{xx} + bt^2 u = 0$$

in the upper half-plane $-\infty < x < \infty,\ 0 < t < \infty$, subject to the initial condition

$$u(x,0) = \varphi(x) \text{ if } -\infty < x < \infty.$$

Here k and b are positive constants and φ is a continuous, absolutely integrable function on $-\infty < x < \infty$.

We take the Fourier transform (with respect to x) of the PDE:

$$\mathcal{F}\left(u_t - ku_{xx} + bt^2 u\right)(\xi) = 0$$

5 pts to here

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2 \mathcal{F}(u)(\xi) + bt^2\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (k\xi^2 + bt^2)\mathcal{F}(u)(\xi) = 0 \longleftarrow \text{a } 1^{st}\text{-order linear ODE in } t \text{ (with param. } \xi)$$

10 An integrating factor is $e^{\int(k\xi^2+bt^2)dt} = e^{k\xi^2 t + bt^3/3}$. Multiplying the ODE through by the integrating factor and integrating the resulting exact expression yields

$$\frac{\partial}{\partial t}\left(e^{k\xi^2 t + bt^3/3}\mathcal{F}(u)(\xi)\right) = e^{k\xi^2 t + bt^3/3}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (k\xi^2 + bt^2)e^{k\xi^2 t + bt^3/3}\mathcal{F}(u)(\xi) = 0$$

15 or $e^{k\xi^2 t + bt^3/3}\mathcal{F}(u)(\xi) = c_1(\xi)$ and hence $\mathcal{F}(u)(\xi) = c_1(\xi)e^{-k\xi^2 t - bt^3/3}$.

Apply entry I in the table of Fourier transforms, $\mathcal{F}\left(\sqrt{2a}\, e^{-a(\cdot)^2}\right)(\xi) = e^{-\frac{\xi^2}{4a}}$,

20 with $\frac{1}{4a} = kt$ to get $\mathcal{F}\left(\frac{1}{\sqrt{2kt}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi) = e^{-k\xi^2 t}$ and apply the initial condition to obtain $\mathcal{F}(\varphi)(\xi) = c_1(\xi)$. Substituting in the equation for $\mathcal{F}(u)(\xi)$

24 gives $\mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)\mathcal{F}\left(\frac{1}{\sqrt{2kt}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)e^{-bt^3/3} = \mathcal{F}(\varphi)(\xi)\mathcal{F}\left(\frac{e^{-bt^3/3}}{\sqrt{2kt}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)$.

29 But $\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(f*g)(\xi)$ so $\mathcal{F}(u)(\xi) = \mathcal{F}\left(\frac{e^{-bt^3/3}}{\sqrt{4k\pi t}}e^{-\frac{(\cdot)^2}{4kt}} * \varphi\right)(\xi)$.

It follows from the inversion theorem that $u(x,t) = \frac{e^{-bt^3/3}}{\sqrt{4k\pi t}}\left(e^{-\frac{(\cdot)^2}{4kt}} * \varphi\right)(x)$;

33 that is,

$$\boxed{u(x,t) = \frac{e^{-bt^3/3}}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\varphi(y)\,dy}$$

for $-\infty < x < \infty$ and $0 < t < \infty$.

2.(33 pts.) Let n be an arbitrary positive integer and consider the heat equation $u_t - Ku_{xx} = 0$ in the upper half-plane $-\infty < x < \infty$, $0 < t < \infty$, subject to the initial condition $u(x,0) = x^n$ if $-\infty < x < \infty$.

(a) Show that this problem has a solution taking the form $u(x,t) = x^n + a_1(t)x^{n-1} + \ldots + a_{n-1}(t)x + a_n(t)$ for suitable functions $a_1(t)$, ..., $a_n(t)$ of t. You may find the binomial formula $(c+d)^n = \sum_{k=0}^{n} \binom{n}{k} c^{n-k} d^k$ useful; recall that the binomial coefficients are given by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

(b) Write out completely the solution in the case $n = 3$; that is, solve the problem completely when the initial condition is $u(x,0) = x^3$ if $-\infty < x < \infty$. You may find the fact $\int_{-\infty}^{\infty} e^{-p^2} p^2 dp = \frac{\sqrt{\pi}}{2}$ useful.

(a) A solution to the problem is given by $u(x,t) = \frac{1}{\sqrt{4K\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Kt}} y^n dy$. Let $p = \frac{y-x}{\sqrt{4Kt}}$; then $dp = \frac{dy}{\sqrt{4Kt}}$ so $u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} (x + p\sqrt{4Kt})^n dp$. Using the binomial formula with $c = x$ and $d = p\sqrt{4Kt}$ yields $u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \sum_{k=0}^{n}$ k p

$$= \sum_{k=0}^{n} x^{n-k} \binom{n}{k} (4Kt)^{k/2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^k dp = \boxed{x^n + \sum_{k=1}^{n} a_k(t) x^{n-k}}$$ where

$$a_k(t) = \left(\frac{n!\, 2^k K^{k/2}}{k!(n-k)!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^k dp \right) t^{k/2} \quad \text{for } k = 1, 2, \ldots, n.$$

(b) When $n = 3$, the solution is $u(x,t) = x^3 + a_1(t)x^2 + a_2(t)x + a_3(t)$ where

$$a_1(t) = \left(\frac{6K^{1/2}}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-p^2} p\, dp}_{\text{odd fcn.}} \right) t^{1/2} = 0, \quad a_2(t) = \left(\frac{12K}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^2 dp \right) t = 6Kt, \text{ and}$$

$$a_3(t) = \left(\frac{8K^{3/2}}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-p^2} p^3 dp}_{\text{odd fcn.}} \right) t^{3/2} = 0. \text{ Therefore } \boxed{u(x,t) = x^3 + 6Ktx}.$$

Check: $u_t - Ku_{xx} = 6Kx - K(6x) \overset{\checkmark}{=} 0$.

$u(x,0) = x^3 + 6K(0)x \overset{\checkmark}{=} x^3$.

26 pts. if either (a) or (b) is perfect.

The other part is worth 7 pts.

3.(33 pts.) Waves in a certain resistant medium satisfy the problem

$$u_{tt} - u_{xx} + u_t \overset{(1)}{=} 0 \text{ if } 0 < x < \pi,\ 0 < t < \infty,$$
$$u(0,t) \overset{(2)}{=} 0 \text{ and } u(\pi,t) \overset{(3)}{=} 0 \text{ if } t \geq 0,$$
$$u(x,0) \overset{(4)}{=} 2\sin(x) \text{ and } u_t(x,0) \overset{(5)}{=} -\sin(x) \text{ if } 0 \leq x \leq \pi.$$

Find a solution of this problem.

Bonus (10 pts.): Show that there is at most one solution to this problem.

We use separation of variables. We seek nontrivial solutions of (1)-(2)-(3) of the form

2 pts. to here $u(x,t) = X(x)T(t)$. Substituting in (1) yields $X(x)T''(t) - X''(x)T(t) + X(x)T'(t) = 0$.

Dividing through by $X(x)T(t)$ and rearranging leads to $-\dfrac{X''(x)}{X(x)} = -\dfrac{(T''(t)+T'(t))}{T(t)} = \text{const.} = \lambda$

Substituting $u(x,t) = X(x)T(t)$ in (2) and (3) gives $X(0)T(t) = 0$ and $X(\pi)T(t) = 0$ if $t \geq 0$.

In order to have nontrivial solutions, it follows that $X(0) = 0 = X(\pi)$. This produces

the coupled system: $\begin{cases} X''(x) + \lambda X(x) \overset{(6)}{=} 0, & X(0) \overset{(7)}{=} 0 \overset{(8)}{=} X(\pi), \\ T''(t) + T'(t) + \lambda T(t) \overset{(9)}{=} 0. \end{cases}$

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The eigenvalue problem (6)-(7)-(8) has Dirichlet boundary conditions. As we saw in

17 Sec. 4.1, the eigenvalues are $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 = \left(\frac{n\pi}{\pi}\right)^2 = n^2$ $(n=1,2,3,\ldots)$ and the eigenfunctions

17 are $X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right) = \sin\left(\frac{n\pi x}{\pi}\right) = \sin(nx)$ $(n=1,2,3,\ldots)$. We next solve (9) with

$\lambda = \lambda_n = n^2$: $T_n''(t) + T_n'(t) + n^2 T_n(t) \overset{(10)}{=} 0$. This is a second-order linear homogeneous

ODE with constant coefficients. Therefore we expect exponential solutions, say $T_n(t) = e^{rt}$

for some constant r. Then $T_n'(t) = re^{rt}$ and $T_n''(t) = r^2 e^{rt}$ so substituting in (10) leads

to $r^2 e^{rt} + re^{rt} + n^2 e^{rt} = 0$ and hence $r^2 + r + n^2 = 0$. The quadratic formula shows

that $r = \dfrac{-1 \pm \sqrt{1-4n^2}}{2} = \dfrac{-1 \pm i\sqrt{4n^2-1}}{2}$. Therefore the general solution of (10) is

20 $T_n(t) = c_n e^{\left(\frac{-1+i\sqrt{4n^2-1}}{2}\right)t} + d_n e^{\left(\frac{-1-i\sqrt{4n^2-1}}{2}\right)t} = c_n e^{-t/2}\left[\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) + i\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right] +$

$d_n e^{-t/2}\left[\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) - i\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right] = e^{-t/2}\left[a_n\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) + b_n\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right].$

2 Consequently $u_n(x,t) = X_n(x)T_n(t) = \sin(nx)e^{-t/2}\left[a_n\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) + b_n\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right]$

solves (1)-(2)-(3) for $n=1,2,3,\ldots$ and arbitrary constants a_n, b_n $(n=1,2,3,\ldots)$.

The superposition principle shows that

-5
$$u(x,t) \overset{(11)}{=} \sum_{n=1}^{N} \sin(nx) e^{-t/2}\left[a_n \cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) + b_n \sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right]$$

solves (1)-(2)-(3) for any positive integer N and any constants $a_1, b_1, \ldots, a_N, b_N$.

We seek a choice of N and constants so that (4) and (5) are satisfied. In particular,

$$2\sin(x) \overset{(4)}{=} u(x,0) \overset{(11)}{=} \sum_{n=1}^{N} a_n \sin(nx) \quad \text{for all } 0 \le x \le \pi.$$ By inspection, we

28 see that $a_1 = 2$ and all other $a_n = 0$ "works". Thus (11) becomes

$$u(x,t) \overset{(12)}{=} 2\sin(x) e^{-t/2} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \sum_{n=1}^{N} b_n \sin(nx) e^{-t/2} \sin\left(\frac{\sqrt{4n^2-1}}{2}t\right).$$

Differentiating with respect to t yields

$$u_t(x,t) \overset{(13)}{=} 2\sin(x)\left[-\frac{1}{2}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$+ \sum_{n=1}^{N} b_n \sin(nx)\left[-\frac{1}{2}e^{-t/2}\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right) + \frac{\sqrt{4n^2-1}}{2}e^{-t/2}\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right].$$

Then we want to choose the b_n's so that

$$-\sin(x) \overset{(5)}{=} u_t(x,0) \overset{(13)}{=} -\sin(x) + \sum_{n=1}^{N} b_n \frac{\sqrt{4n^2-1}}{2}\sin(nx) \quad \text{for all } 0 \le x \le \pi.$$

31 By inspection, we see that $b_n = 0$ for all $n \ge 1$ "works". Therefore

33
$$\boxed{u(x,t) = 2\sin(x)e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)}$$

solves the problem (1)-(2)-(3)-(4)-(5).

<u>Bonus</u>: To show that the problem has at most one solution, suppose that there were two solutions, say $u = u_1(x,t)$ and $u = u_2(x,t)$. Consider

1 pt. to here $v(x,t) = u_1(x,t) - u_2(x,t)$. Then v would be a solution to the system

2
$$\begin{cases} v_{tt} - v_{xx} + v_t \overset{(1)}{=} 0 & \text{if } 0 < x < \pi,\ 0 < t < \infty, \\ v(0,t) \overset{(2)}{=} 0 \text{ and } v(\pi,t) \overset{(3)}{=} 0 & \text{if } t \geq 0, \\ v(x,0) \overset{(4)}{=} 0 \text{ and } v_t(x,0) \overset{(5)}{=} 0 & \text{if } 0 \leq x \leq \pi. \end{cases}$$

The energy of the solution v is given by

$$E(t) = \int_0^{\pi} \left[\tfrac{1}{2} v_t^2(x,t) + \tfrac{1}{2} v_x^2(x,t)\right] dx .$$

We will show that E is a decreasing function of t on $[0,\infty)$. Note that

$$\frac{dE}{dt} = \int_0^{\pi} \frac{\partial}{\partial t}\left[\tfrac{1}{2} v_t^2(x,t) + \tfrac{1}{2} v_x^2(x,t)\right] dx$$

4
$$= \int_0^{\pi} \left[v_t(x,t) v_{tt}(x,t) + v_x(x,t) v_{xt}(x,t)\right] dx .$$

Integrating the second term by parts yields

$$\int_0^{\pi} \overbrace{v_x(x,t)}^{U} \overbrace{v_{xt}(x,t)\,dx}^{dV} = v_x(x,t) v_t(x,t)\Big|_0^{\pi} - \int_0^{\pi} v_t(x,t) v_{xx}(x,t)\,dx$$

5
$$= v_x(\pi,t) v_t(\pi,t) - v_x(0,t) v_t(0,t) - \int_0^{\pi} v_t(x,t) v_{xx}(x,t)\,dx .$$

But (2) and (3) imply $v_t(\pi,t) = \lim\limits_{k\to 0} \dfrac{v(\pi,t+k) - v(\pi,t)}{k} \overset{(2)}{=} \lim\limits_{k\to 0} \dfrac{0}{k} = 0$

and $v_t(0,t) = \lim\limits_{k\to 0} \dfrac{v(0,t+k) - v(0,t)}{k} \overset{(3)}{=} \lim\limits_{k\to 0} \dfrac{0}{k} = 0$. Therefore

6 $\int_0^\pi v_x(x,t)v_{xt}(x,t)dx = -\int_0 v_t(x,t)v_{xx}(x,t)dx$ so substituting in the expression for $\frac{dE}{dt}$ gives

$$\frac{dE}{dt} = \int_0^\pi \left[v_t(x,t)v_{tt}(x,t) - v_t(x,t)v_{xx}(x,t)\right]dx = \int_0^\pi v_t(x,t)\left[v_{tt}(x,t) - v_{xx}(x,t)\right]dx.$$

Applying ① then yields

7 $$\frac{dE}{dt} = \int_0^\pi v_t(x,t)\left[-v_t(x,t)\right]dx = -\int_0^\pi v_t^2(x,t)dx \leq 0$$

so E is a decreasing function on $[0,\infty)$. That is, for all $t>0$,

8 $$0 \leq E(t) \leq E(0) = \int_0^\pi \left[\tfrac{1}{2}v_t^2(x,0) + \tfrac{1}{2}v_x^2(x,0)\right]dx.$$

But $v_t(x,0) = 0$ for $0 \leq x \leq \pi$ by ⑤ while ④ implies

$$v_x(x,0) = \lim_{h\to 0} \frac{v(x+h,0) - v(x,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

for all $0 \leq x \leq \pi$. Therefore $E(0) = 0$ so

9 $$0 = E(t) = \int_0^\pi \left[\tfrac{1}{2}v_t^2(x,t) + \tfrac{1}{2}v_x^2(x,t)\right]dx$$

for all $t \geq 0$. The vanishing theorem then implies $\frac{1}{2}v_t^2(x,t) + \frac{1}{2}v_x^2(x,t) = 0$ for all $0 \leq x \leq \pi$ and each $t \geq 0$, so $v_t(x,t) = 0 = v_x(x,t)$ and hence $v(x,t) = \text{constant}$ for $0 \leq x \leq \pi$ and $t \geq 0$. But $v(x,0) \overset{④}{=} 0$ for all $0 \leq x \leq \pi$

10 so $v(x,t) = 0$ for all $0 \leq x \leq \pi$ and $t \geq 0$. That is, $u_1(x,t) = u_2(x,t)$ for $0 \leq x \leq \pi$ and $t \geq 0$, so there is at most one solution to the problem in 3.

A Brief Table of Fourier Transforms

$f(x)$	$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx$
A. $\begin{cases}1 & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$
B. $\begin{cases}1 & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C. $\frac{1}{x^2+a^2}\quad(a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\xi\rvert}}{a}$
D. $\begin{cases}x & \text{if } 0<x\le b,\\ 2b-x & \text{if } b<x<2b,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E. $\begin{cases}e^{-ax} & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{1}{(a+i\xi)\sqrt{2\pi}}$
F. $\begin{cases}e^{ax} & \text{if } b<x<c,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G. $\begin{cases}e^{iax} & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$
H. $\begin{cases}e^{iax} & \text{if } c<x<d.\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I. $e^{-ax^2}\quad(a>0)$	$\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J. $\frac{\sin(ax)}{x}\quad(a>0)$	$\begin{cases}0 & \text{if } \lvert\xi\rvert\ge a,\\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert<a.\end{cases}$

Math 325

Exam 2

Spring 2011

number taking exam: 24

mean: 62.0 median: 65.0

standard deviation: 25.5

Distribution of Scores:

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87–100	A	A	6
73 – 86	B	B	4
60 – 72	C	B	3
50 – 59	C	C	3
0 – 49	F	D	8