Mathematics 325 Exam II Summer 2011 Name: Dr. Grow

1.(34 pts.) Find the solution of

$$u_{tt} - u_{xx} \overset{(1)}{=} 0 \quad \text{if } 0 < x < \pi/2,\ 0 < t < \infty,$$

satisfying the boundary conditions

$$u_x(0,t) \overset{(2)}{=} 0 \quad \text{and} \quad u(\pi/2,t) \overset{(3)}{=} 0 \quad \text{if } t \ge 0$$

(i.e. Neumann at the left end and Dirichlet at the right end), and the initial conditions

$$u(x,0) \overset{(5)}{=} \cos^3(x) \quad \text{and} \quad u_t(x,0) \overset{(4)}{=} 0 \quad \text{if } 0 \le x \le \pi/2.$$

Hint: You may find useful the identity $\frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos(\theta) = \cos^3(\theta)$.

Bonus (10 pts.): Show that there is at most one solution to the above problem.

We use separation of variables. We seek nontrivial solutions to (1)-(2)-(3)-(4) of the form

(*) $u(x,t) = X(x)T(t)$. Substituting from (*) into (1) yields $X(x)T''(t) - X''(x)T(t) = 0$, or equivalently $\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda$. Substituting from (*) into (2) yields $X'(0)T(t) = 0$ for all $t \ge 0$. In order for the solution (*) to be nontrivial we need $X'(0) = 0$. Substituting from (*) into (3) yields $X(\frac{\pi}{2})T(t) = 0$ for all $t \ge 0$, whence $X(\pi/2) = 0$ for nontrivial solutions (*). Substituting from (*) into (4) gives $X(x)T'(0) = 0$ for all $0 \le x \le \pi/2$. It follows that $T'(0) = 0$ if the solution (*) is to be nontrivial. We collect these constraints in the following coupled system of ODEs and BCs:

$$X''(x) + \lambda X(x) \overset{(6)}{=} 0, \quad X'(0) \overset{(7)}{=} 0, \quad X\left(\tfrac{\pi}{2}\right) \overset{(8)}{=} 0,$$

$$T''(t) + \lambda T(t) \overset{(9)}{=} 0, \quad T'(0) \overset{(10)}{=} 0.$$

13 pts to here

We next solve the eigenvalue problem (6)-(7)-(8). We assume that the eigenvalues λ are real. This leads to three cases: $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$.

<u>Case $\lambda > 0$</u>, say $\lambda = k^2$ where $k > 0$:

Then (6) becomes $X''(x) + k^2 X(x) = 0$, which has general solution $X(x) = c_1\cos(kx) + c_2\sin(kx)$. Note that $X'(x) = -kc_1\sin(kx) + kc_2\cos(kx)$ so $0 \overset{(7)}{=} X'(0) = -kc_1\sin(0) + kc_2\cos(0)$ and it follows that $c_2 = 0$. Then $0 \overset{(8)}{=} X(\pi/2) = c_1\cos(k\frac{\pi}{2})$ and hence $k = 1, 3, 5, \ldots$ for nontrivial solutions. That is, the eigenvalues and eigenfunctions in this case are

$$\lambda_n = k_n^2 = (2n-1)^2 \quad \text{and} \quad X_n(x) = \cos(k_n x) = \cos((2n-1)x) \quad (n = 1, 2, 3, \ldots)$$

18 pts to here

Case $\lambda = 0$: Then (6) becomes $X''(x) = 0$, which has general solution $X(x) = c_1 x + c_2$. Note that $X'(x) = c_1$ so $0 \overset{(7)}{=} X'(0) = c_1$. Then $0 \overset{(8)}{=} X(\pi/2) = c_2$. That is, all solutions of (6)-(7)-(8) are trivial when $\lambda = 0$, so zero is not an eigenvalue.

3 pts to here.

Case $\lambda < 0$, say $\lambda = -k^2$ where $k > 0$: Then (6) becomes $X''(x) - k^2 X(x) = 0$, which has general solution $X(x) = c_1\cosh(kx) + c_2\sinh(kx)$. Observe that $X'(x) = kc_1\sinh(kx) + kc_2\cosh(kx)$ so $0 \overset{(7)}{=} X'(0) = kc_1\sinh(0) + kc_2\cosh(0)$ and it follows that $c_2 = 0$. Then $0 \overset{(8)}{=} X(\pi/2) = c_1\cosh(k\pi/2)$ so $c_1 = 0$. Since all solutions to (6)-(7)-(8) are trivial when $\lambda < 0$, there are no negative eigenvalues.

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We next seek solutions to (9)-(10) for the eigenvalues $\lambda = \lambda_n = (2n-1)^2$ $(n = 1, 2, 3, \ldots)$. Then (9) becomes $T_n''(t) + (2n-1)^2 T_n(t) = 0$, which has $T_n(t) = c_1\cos((2n-1)t) + c_2\sin((2n-1)t)$ as the general solution. Since $T_n'(t) = -(2n-1)c_1\sin((2n-1)t) + (2n-1)c_2\cos((2n-1)t)$ it follows that $0 \overset{(10)}{=} T_n'(0) = -(2n-1)c_1\sin(0) + (2n-1)c_2\cos(0)$ and thus $c_2 = 0$. That is, $T_n(t) = \cos((2n-1)t)$ up to a constant factor. Therefore the nontrivial solutions to (1)-(2)-(3)-(4) of the form (*) are $u_n(x,t) = X_n(x)T_n(t) = \cos((2n-1)x)\cos((2n-1)t)$ $(n = 1, 2, 3, \ldots)$. The superposition principle implies that

26 pts to here.

$$(**) \qquad u(x,t) = \sum_{n=1}^{N} c_n \cos((2n-1)x)\cos((2n-1)t)$$

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solves (1)-(2)-(3)-(4) for any positive integer N and any choice of constants $c_1, c_2, \ldots, c_N$. We want to choose N and the c_n so that (5) is satisfied. That is,

$$\frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x) \overset{\text{(useful identity)}}{=} \cos^3(x) \overset{(5)}{=} u(x,0) = \sum_{n=1}^{N} c_n\cos((2n-1)x) \quad \text{for all } 0 \le x \le \pi/2$$

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By inspection, we may take $N = 2$, $c_1 = \frac{3}{4}$, and $c_2 = \frac{1}{4}$. Thus

34 pts to here.

$$\boxed{u(x,t) = \frac{3}{4}\cos(x)\cos(t) + \frac{1}{4}\cos(3x)\cos(3t)}$$

solves (1)-(2)-(3)-(4)-(5).

Bonus: Suppose that $v = v(x,t)$ is another solution to (1)-(2)-(3)-(4)-(5). Then

pts. to here.

$$w(x,t) = \frac{3}{4}\cos(x)\cos(t) + \frac{1}{4}\cos(3x)\cos(3t) - v(x,t) \text{ solves:}$$

4 pts. to here.

$$\begin{cases} w_{tt} - w_{xx} \overset{(11)}{=} 0 & \text{if } 0 < x < \pi/2,\ 0 < t < \infty, \\ w_x(0,t) \overset{(12)}{=} 0 \overset{(13)}{=} w(\pi/2, t) & \text{if } t \geq 0, \\ w(x,0) \overset{(14)}{=} 0 \overset{(15)}{=} w_t(x,0) & \text{if } 0 \leq x \leq \pi/2. \end{cases}$$

5 pts. to here.

Consider the enery function of w: $E(t) = \int_0^{\pi/2} \left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx$ for $t \geq 0$.

Then $\frac{dE}{dt} = \int_0^{\pi/2} \frac{\partial}{\partial t}\left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx = \int_0^{\pi/2}\left[w_t(x,t)w_{tt}(x,t) + w_x(x,t)w_{tx}(x,t)\right]dx$

$= \int_0^{\pi/2} w_t w_{tt}\,dx + w_x(x,t)w_t(x,t)\Big|_{x=0}^{\pi/2} - \int_0^{\pi/2} w_t(x,t)w_{xx}(x,t)\,dx$. But (13) implies $w_t(\pi/2, t) = 0$

and (12) then shows that $w_x(x,t)w_t(x,t)\Big|_{x=0}^{\pi/2} = 0$ so $\frac{dE}{dt} = \int_0^{\pi/2} w_t(x,t)\left[w_{tt}(x,t) - w_{xx}(x,t)\right]dx$

8 pts to here. $= 0$ by (11). That is, the energy of w is a constant function of $t \geq 0$. But

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$$E(0) = \int_0^{\pi/2}\left[\frac{1}{2}w_t^2(x,0) + \frac{1}{2}w_x^2(x,0)\right]dx = 0 \text{ by (14) and (15) so } E(t) = 0 \text{ for all } t \geq 0.$$

The vanishing theorem then implies $\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t) = 0$ for all $0 \leq x \leq \pi/2$ and $t \geq 0$. Hence $w_t(x,t) = 0 = w_x(x,t)$ so $w = w(x,t)$ is a constant function on $0 \leq x \leq \pi/2$, $0 \leq t < \infty$. But (14) then shows that $w(x,t) = 0$ for all $0 \leq x \leq \pi/2$ and $t \geq 0$. That is, the solution to (1)-(2)-(3)-(4)-(5) is unique:

10 pts to here.

$$v(x,t) = \frac{3}{4}\cos(x)\cos(t) + \frac{1}{4}\cos(3x)\cos(3t).$$

2.(33 pts.) Solve $u_t - u_{xx} = 0$ in the upper half-plane $-\infty < x < \infty$, $0 < t < \infty$, subject to the initial condition $u(x,0) = e^{-x^2}$ if $-\infty < x < \infty$.

A solution to $u_t - ku_{xx} = 0$ in the upper half-plane satisfying $u(x,0) = \varphi(x)$ if $-\infty < x < \infty$ is given by

7 pts. to here.

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy.$$

In our case $k=1$ and $\varphi(x) = e^{-x^2}$ so

14 pts. to here

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \cdot e^{-y^2} dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{[(x-y)^2 + 4ty^2]}{4t}} dy.$$

We need to complete the square in y in the exponent of the integrand:

$$\begin{aligned}
(x-y)^2 + 4ty^2 &= x^2 - 2xy + (1+4t)y^2 \\
&= x^2 + \left(\sqrt{1+4t}\,y\right)^2 - 2\left(\frac{x}{\sqrt{1+4t}}\right)\left(\sqrt{1+4t}\,y\right) + \left(\frac{x}{\sqrt{1+4t}}\right)^2 - \left(\frac{x}{\sqrt{1+4t}}\right)^2 \\
&= x^2 - \frac{x^2}{1+4t} + \left(\sqrt{1+4t}\,y - \frac{x}{\sqrt{1+4t}}\right)^2 \\
&= \frac{4tx^2}{1+4t} + \left(\sqrt{1+4t}\,y - \frac{x}{\sqrt{1+4t}}\right)^2.
\end{aligned}$$

24 pts. to here.

Therefore

27 pts. to here.

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{1+4t}} \cdot e^{-\frac{\left(\sqrt{1+4t}\,y - \frac{x}{\sqrt{1+4t}}\right)^2}{4t}} dy.$$

Let $p = \dfrac{\sqrt{1+4t}\,y - \frac{x}{\sqrt{1+4t}}}{\sqrt{4t}}$. Then $dp = \dfrac{\sqrt{1+4t}}{\sqrt{4t}} dy$ and as $y \to \pm\infty$, $p \to \pm\infty$, so

$$u(x,t) = \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{x^2}{1+4t}} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{\sqrt{1+4t}} = \frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}} \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp = \boxed{\frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}}.$$

31 pts. to here.

33 pts. to here.

3.(33 pts.) Use the method of Fourier transforms to solve $u_{tt}+u_{xx} \overset{(1)}{=} 0$ in the upper half-plane $-\infty < x < \infty,\ 0 < t < \infty$, subject to the initial condition

$$u(x,0) \overset{(2)}{=} \begin{cases} 5 & \text{if } 0<x<2, \\ 0 & \text{otherwise,} \end{cases}$$

and the decay condition $\lim_{t\to\infty} u(x,t) \overset{(3)}{=} 0$ for each x in $(-\infty,\infty)$. Note: For full credit, do not leave any unevaluated integrals in your final answer.

Let $u = u(x,t)$ be a solution to the problem. Then $u_{tt}(x,t) + u_{xx}(x,t) = 0$ for all $-\infty < x < \infty$, $0 < t < \infty$, so taking the Fourier transform of both sides of this identity yields

$$\frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi) + (i\xi)^2\mathcal{F}(u)(\xi) = \mathcal{F}\left(u_{tt}+u_{xx}\right)(\xi) = \mathcal{F}(0)(\xi) = 0,$$

or equivalently

$$\frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi) - \xi^2\mathcal{F}(u)(\xi) = 0.$$

4 pts to here

The general solution of this linear, second-order, homogeneous ordinary differential equation in t (with parameter ξ) is

$$\mathcal{F}(u)(\xi) = c_1(\xi)e^{\xi t} + c_2(\xi)e^{-\xi t}.$$

6 pts to here.

Suppose $\xi > 0$. Then applying (3) yields

$$0 = \mathcal{F}\left(\lim_{t\to\infty} u(x,t)\right)(\xi) = \lim_{t\to\infty}\mathcal{F}(u)(\xi) = \lim_{t\to\infty}\left(c_1(\xi)e^{\xi t} + c_2(\xi)e^{-\xi t}\right) = \lim_{t\to\infty} c_1(\xi)e^{\xi t}.$$

so we <u>must</u> have $c_1(\xi) = 0$ for $\xi > 0$. Arguing similarly in the case when $\xi < 0$, (3) gives

$$0 = \mathcal{F}\left(\lim_{t\to\infty} u(x,t)\right)(\xi) = \lim_{t\to\infty}\mathcal{F}(u)(\xi) = \lim_{t\to\infty}\left(c_1(\xi)e^{\xi t} + c_2(\xi)e^{-\xi t}\right) = \lim_{t\to\infty} c_2(\xi)e^{-\xi t},$$

so we <u>must</u> have $c_2(\xi) = 0$ for $\xi < 0$. Thus

$$\mathcal{F}(u)(\xi) = \begin{cases} c_2(\xi)e^{-\xi t} & \text{if } \xi > 0, \\ c_1(\xi)e^{\xi t} & \text{if } \xi < 0, \end{cases} = A(\xi)e^{-|\xi|t}.$$

10 pts here.

Using the notation $\chi_{(0,2)}(x) = \begin{cases} 1 & \text{if } 0<x<2, \\ 0 & \text{otherwise}, \end{cases}$

we have by ② that

19 pts. to here.

$$5\mathcal{F}(\chi_{(0,2)})(\xi) = \mathcal{F}(u)(\xi)\Big|_{t=0} = A(\xi)e^{-|\xi|t}\Big|_{t=0} = A(\xi).$$

Therefore $\mathcal{F}(u)(\xi) = 5\mathcal{F}(\chi_{(0,2)})(\xi)\, e^{-|\xi|t}$. By entry C in the brief table of Fourier transforms with $a=t$ we have

$$\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\ \frac{t}{(\cdot)^2+t^2}\right)(\xi) = e^{-|\xi|t}$$

so

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$$\mathcal{F}(u)(\xi) = 5\mathcal{F}(\chi_{(0,2)})(\xi)\,\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\ \frac{t}{(\cdot)^2+t^2}\right)(\xi).$$

The convolution property $\mathcal{F}(f*g)(\xi) = \sqrt{2\pi}\,\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)$ then implies

$$\mathcal{F}(u)(\xi) = \frac{5}{\sqrt{2\pi}}\mathcal{F}\left(\chi_{(0,2)} * \sqrt{\frac{2}{\pi}}\ \frac{t}{(\cdot)^2+t^2}\right)(\xi)$$

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$$= \mathcal{F}\left(\frac{5}{\pi}\chi_{(0,2)} * \frac{t}{(\cdot)^2+t^2}\right)(\xi).$$

The Fourier inversion theorem then implies that for $-\infty<x<\infty$, $0<t<\infty$,

$$u(x,t) = \frac{5}{\pi}\left(\chi_{(0,2)} * \frac{t}{(\cdot)^2+t^2}\right)(x)$$

29 pts. to here.

$$= \frac{5}{\pi}\int_{-\infty}^{\infty} \frac{t}{(x-y)^2+t^2}\,\chi_{(0,2)}(y)\,dy$$

$$= \frac{5}{\pi}\int_0^2 \frac{1}{\left(\frac{x-y}{t}\right)^2+1}\,dy/t$$

$$= -\frac{5}{\pi}\,\mathrm{Arctan}\left(\frac{x-y}{t}\right)\Big|_{y=0}^{2}$$

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$$= \boxed{\frac{5}{\pi}\left[\mathrm{Arctan}\left(\frac{x}{t}\right) - \mathrm{Arctan}\left(\frac{x-2}{t}\right)\right]}.$$

A Brief Table of Fourier Transforms

$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A. $\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B. $\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C. $\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\xi\rvert}}{a}$
D. $\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E. $\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
F. $\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G. $\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi - a))}{\xi - a}$
H. $\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{i(\xi - a)\sqrt{2\pi}}$
I. $e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J. $\frac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert \ge a, \\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert < a. \end{cases}$

Math 325

Exam II

Summer 2011

number taking exam : 24

mean : 72.2

standard deviation : 23.1

Distribution of Scores :

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87–100	A	A	11
73 – 86	B	B	0
60 – 72	C	B	5
50 – 59	C	C	2
0 – 49	F	D	6