Mathematics 325 Exam II Name: Dr. Grow

Summer 2012

1.(35 pts.) Solve $\dfrac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}=0$ in the upper half-plane $-\infty<x<\infty,\ 0<t<\infty$, subject to the initial condition $u(x,0)=e^{3x}$ for all real x.

We would like to apply $u(x,t)=\dfrac{1}{\sqrt{4k\pi t}}\displaystyle\int_{-\infty}^{\infty}\varphi(y)\,e^{-\frac{(x-y)^2}{4kt}}\,dy$ (3 pts.) with $k=1$ (1 pt.) and $\varphi(y)=e^{3y}$ (1 pt.). However the function φ is not bounded on $(-\infty,\infty)$ so the formula does not apply. Therefore we will need to check our answer at the end. Thus,

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{3y}\cdot e^{-\frac{(x-y)^2}{4t}}\,dy=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{\frac{12yt-(x-y)^2}{4t}}\,dy.$$

3 pts. 2 pts.

We need to complete the square in the variable y in the integrand's exponent:

$$12yt-(x-y)^2=12yt-x^2+2xy-y^2=-\left[y^2-2y(x+6t)+(x+6t)^2\right]+(x+6t)^2-x^2$$
$$=-\left[y-(x+6t)\right]^2+\cancel{x^2}+12xt+36t^2-\cancel{x^2}=-\left[y-(x+6t)\right]^2+4t(3x+9t).$$

Therefore

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{\frac{-[y-(x+6t)]^2+4t(3x+9t)}{4t}}\,dy=\frac{e^{3x+9t}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{-\frac{[y-(x+6t)]^2}{4t}}\,dy.$$

Let $p=\dfrac{y-(x+6t)}{\sqrt{4t}}$. Then $dp=\dfrac{dy}{\sqrt{4t}}$ so $u(x,t)=\dfrac{e^{3x+9t}}{\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty}e^{-p^2}\,dp$. But $\displaystyle\int_{-\infty}^{\infty}e^{-p^2}dp=\sqrt{\pi}$ so $\boxed{u(x,t)=e^{3x+9t}.}$

Check: $u_t-u_{xx}=9e^{3x+9t}-9e^{3x+9t}\overset{\checkmark}{=}0$ for all real x and t.

$u(x,0)=e^{3x+0}\overset{\checkmark}{=}e^{3x}$ for all real x.

2.(35 pts.) Use Fourier transform methods to derive a formula for the solution to $u_{xx}+u_{yy}=0$ if $-\infty<x<\infty,\ 0<y<\infty$, subject to the boundary condition $u(x,0)=\varphi(x)$ if $-\infty<x<\infty$ and the decay condition $\lim_{y\to\infty} u(x,y)=0$ for each real number x.

In this problem, let $\mathcal{F}$ denote the Fourier transform operator with respect to the variable x. I.e. if $f=f(x,y)$ then $\mathcal{F}(f)(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,y)e^{-i\xi x}\,dx$. Let $u=u(x,y)$ be a solution to the problem above so $u_{xx}(x,y)+u_{yy}(x,y)=0$ for all $-\infty<x<\infty,\ 0<y<\infty$. Taking the Fourier transform of both sides of this identity gives

2pts.

$$0=\mathcal{F}(u_{xx}+u_{yy})(\xi)=\mathcal{F}(u_{xx})(\xi)+\mathcal{F}(u_{yy})(\xi)=(i\xi)^2\mathcal{F}(u)(\xi)+\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi)$$

so $\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi)-\xi^2\mathcal{F}(u)(\xi)=0$. The general solution of this second order ODE is

7pts.

$\mathcal{F}(u)(\xi)=c_1(\xi)e^{\xi y}+c_2(\xi)e^{-\xi y}$. The decay condition $\lim_{y\to\infty} u(x,y)=0$ for all real x implies

4pts.

$$0=\lim_{y\to\infty}\mathcal{F}(u)(\xi)=\lim_{y\to\infty}\left(c_1(\xi)e^{\xi y}+c_2(\xi)e^{-\xi y}\right)$$

so $c_1(\xi)=0$ if $\xi>0$ and $c_2(\xi)=0$ if $\xi<0$. That is,

$$\mathcal{F}(u)(\xi)=\begin{Bmatrix} c_2(\xi)e^{-\xi y} & \text{if } \xi>0 \\ c_1(\xi)e^{\xi y} & \text{if } \xi<0\end{Bmatrix}=c(\xi)e^{-|\xi|y}.$$

5pts.

Applying the boundary condition $u(x,0)=\varphi(x)$ for all real x leads to

$$\mathcal{F}(\varphi)(\xi)=\mathcal{F}(u)(\xi)\Big|_{y=0}=c(\xi)e^{-|\xi|y}\Big|_{y=0}=c(\xi)\quad\text{for all real } \xi.$$

2pts.

Thus $\mathcal{F}(u)(\xi)=\mathcal{F}(\varphi)(\xi)\cdot e^{-|\xi|y}\overset{\text{F.T. table, entry C. with } a=y}{=}\mathcal{F}(\varphi)(\xi)\,\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}\right)(\xi)$.

2pts.

Using the convolution identity $\mathcal{F}(f*g)(\xi)=\sqrt{2\pi}\,\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)$ gives

$$\mathcal{F}(u)(\xi)=\frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi*\sqrt{\frac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}\right)(\xi)=\mathcal{F}\left(\varphi*\frac{1}{\pi}\frac{y}{(\cdot)^2+y^2}\right)(\xi)$$

The inversion theorem then yields

$$\boxed{u(x,y)=\left(\varphi*\frac{1}{\pi}\frac{y}{(\cdot)^2+y^2}\right)(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(z)\,y}{(x-z)^2+y^2}\,dz.}$$

3.(30 pts.) (a) Let $E(t)=\frac{1}{2}\int_0^1\left\{\left[u_t(x,t)\right]^2+\left[u_{xx}(x,t)\right]^2\right\}dx$ denote the energy at time $t\geq 0$ of a solution $u=u(x,t)$ to the problem:

$$u_{tt}+u_{xxxx}=0 \text{ if } 0<t<\infty,\ 0<x<1,$$
$$u_x(0,t)=0=u_x(1,t) \text{ and } u_{xxx}(0,t)=0=u_{xxx}(1,t) \text{ if } t\geq 0,$$
$$u(x,0)=0=u_t(x,0) \text{ if } 0\leq x\leq 1.$$

Show that the energy function is a constant for $t\geq 0$.

(b) Use the result of part (a) to show that there is at most one solution to the problem:

$$u_{tt}+u_{xxxx}=f(x,t) \text{ if } 0<t<\infty,\ 0<x<1,$$
$$u_x(0,t)=g(t),\ u_x(1,t)=h(t),\ u_{xxx}(0,t)=m(t), \text{ and } u_{xxx}(1,t)=n(t) \text{ if } t\geq 0,$$
$$u(x,0)=r(x) \text{ and } u_t(x,0)=s(x) \text{ if } 0\leq x\leq 1.$$

[Note: Even if you are not able to solve part (a), you may use the result of part (a) in answering part (b).]

(a) Note that $\frac{dE}{dt}=\frac{1}{2}\frac{d}{dt}\int_0^1\left[u_t^2(x,t)+u_{xx}^2(x,t)\right]dx=\frac{1}{2}\int_0^1\frac{\partial}{\partial t}\left[u_t^2(x,t)+u_{xx}^2(x,t)\right]dx$

$=\int_0^1\left[u_t(x,t)u_{tt}(x,t)+u_{xx}(x,t)u_{xxt}(x,t)\right]dx.$ (*) Two integrations by parts gives

$$\int_0^1 \overbrace{u_{xx}}^{v}\,\overbrace{u_{xxt}\,dx}^{dv}=u_{xx}u_{xt}\Big|_{x=0}^{1}-\int_0^1\overbrace{u_{xxx}}^{v}\,\overbrace{u_{xt}\,dx}^{dv}=\left(u_{xx}u_{xt}-u_{xxx}u_t\right)\Big|_{x=0}^{1}+\int_0^1 u_{xxxx}u_t\,dx$$

Differentiating the boundary conditions $u_x(0,t)=0=u_x(1,t)$ for $t\geq 0$ with respect to t holding x fixed gives $u_{xt}(0,t)=0=u_{xt}(1,t)$ for $t\geq 0$. This, together with the boundary conditions $u_{xxx}(0,t)=0=u_{xxx}(1,t)$ for $t\geq 0$, imply

$$\left(u_{xx}u_{xt}-u_{xxx}u_t\right)\Big|_{x=0}^{1}=u_{xx}(1,t)\overbrace{u_{xt}(1,t)}^{0}-\overbrace{u_{xxx}(1,t)}^{0}u_t(1,t)-u_{xx}(0,t)\overbrace{u_{xt}(0,t)}^{0}+\overbrace{u_{xxx}(0,t)}^{0}u_t(0,t)$$
$$=0 \quad \text{for all } t\geq 0.$$

Therefore $\int_0^1 u_{xx}u_{xxt}\,dx=\int_0^1 u_{xxxx}u_t\,dx$, so substituting in the identity (*) gives

$$\frac{dE}{dt}=\int_0^1\left[u_t(x,t)u_{tt}(x,t)+u_{xxxx}(x,t)u_t(x,t)\right]dx$$
$$=\int_0^1 u_t(x,t)\Big[\underbrace{u_{tt}(x,t)+u_{xxxx}(x,t)}_{0 \text{ from the PDE}}\Big]dx$$
$$=0 \quad \text{for all } t\geq 0.$$

That is, $\boxed{E(t)=E(0)=\text{constant for all } t\geq 0.}$

(OVER for (b))

(b) Suppose that $u = u_1(x,t)$ and $u = u_2(x,t)$ are solutions to the problem in (b). Then $u(x,t) = u_1(x,t) - u_2(x,t)$ solves the associated homogeneous problem

2 pts.

$$\begin{cases} u_{tt} + u_{xxxx} = 0 & \text{if } 0 < x < 1,\ 0 < t < \infty, \quad \text{2 pts.} \\ u_x(0,t) = 0 = u_x(1,t) \text{ and } u_{xxx}(0,t) = 0 = u_{xxx}(1,t) & \text{if } t \geq 0, \quad \text{2 pts.} \\ u(x,0) = 0 = u_t(x,0) & \text{if } 0 \leq x \leq 1. \quad \text{2 pts.} \end{cases}$$

By part (a), for each fixed $t \geq 0$,

$$(\dagger) \quad \frac{1}{2}\int_0^1 \left[u_t^2(x,t) + u_{xx}^2(x,t)\right]dx = E(t) = E(0) = \frac{1}{2}\int_0^1 \left[u_t^2(x,0) + u_{xx}^2(x,0)\right]dx. \quad \text{2 pts.}$$

The initial condition $u_t(x,0) = 0$ for all $0 \leq x \leq 1$ and the identity $u_{xx}(x,0) = 0$ 1 pt. for all $0 \leq x \leq 1$, obtained by differentiating twice the initial condition $u(x,0) = 0$ if $0 \leq x \leq 1$, imply that $E(0) = 0$. The vanishing theorem applied to ($\dagger$) then implies $u_t^2(x,t) + u_{xx}^2(x,t) = 0$ 1 pt. for all $0 \leq x \leq 1$ and each fixed $t \geq 0$. Thus $u_t(x,t) = 0 = u_{xx}(x,t)$ for all $0 \leq x \leq 1$ and all $t \geq 0$. In particular,

$$(\dagger\dagger) \quad u(x,t) - u(x,0) = \int_0^t u_t(x,\tau)\,d\tau = \int_0^t 0\,d\tau = 0 \quad \text{2 pts.}$$

for all $t \geq 0$ and each fixed x in $[0,1]$. But $u(x,0) = 0$ for all $0 \leq x \leq 1$ so ($\dagger\dagger$) implies $u(x,t) = 0$ 1 pt. for all $0 \leq x \leq 1$ and all $t \geq 0$. That is,

$$\boxed{u_1(x,t) = u_2(x,t)}$$

for all $0 \leq x \leq 1$ and $0 \leq t$ so solutions to (b) are unique.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$
A.	$\begin{cases}1 & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases}1 & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2+a^2} \quad (a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases}x & \text{if } 0<x\le b,\\ 2b-x & \text{if } b<x<2b,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases}e^{-ax} & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases}e^{ax} & \text{if } b<x<c,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases}e^{iax} & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases}e^{iax} & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a>0)$	$\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a>0)$	$\begin{cases}0 & \text{if } \lvert\xi\rvert\ge a,\\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert<a.\end{cases}$

Math 325

Exam II

Summer 2012

$n = 16$

mean = 59.1

median = 62

standard deviation = 27.9

Distribution of Scores

Range	Graduate Grade	Undergraduate Grade	Frequency
87-100	A	A	4
73 - 86	B	B	1
60 - 72	C	B	3
50 - 59	C	C	2
0 - 49	F	D	6