Mathematics 325 Exam III Name: Dr. Grow
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1.(30 pts.) Consider the operator $T = -\dfrac{d^2}{dx^2}$ on $V = \{\varphi \in C^2[0,1]:\ \varphi'(0) = 0,\ \varphi(1) = 0\}$.

8 (a) Show that T is a symmetric operator on V; i.e. show that $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all f and g in V.

6 (b) Are the eigenvalues of T on V real? Justify your answer.

6 (c) Are the eigenfunctions of T on V corresponding to distinct eigenvalues orthogonal on $(0,1)$? Justify your answer.

10 (d) Are the eigenvalues of T on V nonnegative? Justify your answer.

(a) Let f and g belong to V. Integrating by parts twice gives

$$\langle Tf, g\rangle \overset{1}{=} \int_0^1 -f''(x)\overline{g(x)}\,dx \overset{4}{=} \left[-\overline{g(x)}f'(x) + \overline{g'(x)}f(x)\right]\Big|_{x=0}^{1} - \int_0^1 f(x)\overline{g''(x)}\,dx.$$ But

$g(1) = 0$, $f(1) = 0$, $g'(0) = 0$, and $f'(0) = 0$ so the boundary terms all evaluate to zero. Thus $\langle Tf, g\rangle \overset{1}{=} \langle f, Tg\rangle$.

(b) Since $T: V \to C[0,1]$ is a symmetric operator, Theorem 2 in Section 5.3 shows that the eigenvalues of T are real.

(c) Since $T: V \to C[0,1]$ is a symmetric operator, Theorem 1 in Section 5.3 shows that the eigenfunctions of T corresponding to distinct eigenvalues are othogonal on $(0,1)$.

(d) Let λ be an (necessarily real) eigenvalue of $T: V \to C[0,1]$ and let $\varphi \in V$ be an eigenfunction of T corresponding to λ. Then

$$\lambda\langle\varphi,\varphi\rangle \overset{1}{=} \langle\lambda\varphi,\varphi\rangle \overset{1}{=} \langle T\varphi,\varphi\rangle \overset{1}{=} \int_0^1 -\varphi''(x)\overline{\varphi(x)}\,dx.$$

Integrating by parts yields

$$\lambda\langle\varphi,\varphi\rangle \overset{2}{=} -\varphi'(x)\overline{\varphi(x)}\Big|_{x=0}^{1} + \int_0^1 \varphi'(x)\overline{\varphi'(x)}\,dx.$$

But $\varphi(1) = 0 = \varphi'(0)$ since $\varphi \in V$ so the boundary terms vanish and so

$$\lambda\langle\varphi,\varphi\rangle = \int_0^1 |\varphi'(x)|^2\,dx.$$

But $\langle\varphi,\varphi\rangle > 0$ and $\int_0^1 |\varphi'(x)|^2\,dx \geq 0$ so $\lambda \geq 0$. That is, the eigenvalues of T are nonnegative.

2.(35 pts.) On this problem, you may assume that the set of functions

$$\Phi = \{\varphi_n(x) = \cos((2n+1)\pi x/2) : n = 0,1,2,\dots\}$$

is orthogonal on the interval $[0,1]$. The identity $\cos^2(A) = (1+\cos(2A))/2$ may also prove useful.

6 (a) Show that the Fourier series of the function $f(x) = 1-x^2$ with respect to Φ on the interval $[0,1]$ is

$$\sum_{n=0}^{\infty} \frac{32(-1)^n \cos((2n+1)\pi x/2)}{(2n+1)^3 \pi^3}.$$

6 (b) Does the Fourier series of f with respect to Φ converge uniformly to f on $[0,1]$? Justify your answer.

6 (c) Does the Fourier series of f with respect to Φ converge pointwise to f on $[0,1]$? Justify your answer.

5 (d) Does the Fourier series of f with respect to Φ converge in the L^2 – sense to f on $[0,1]$? Justify your answer.

6 (e) Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$.

6 (f) Find the sum of the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6}$.

(a) $f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x)$ where $c_n \overset{(1)}{=} \frac{\langle f,\varphi_n\rangle}{\langle \varphi_n,\varphi_n\rangle}$ for $n = 0,1,2,\dots$ In our case, $\langle \varphi_n,\varphi_n\rangle = \int_0^1 \cos^2((n+\tfrac{1}{2})\pi x)\,dx = \frac{1}{2}\int_0^1 (1+\cos((2n+1)\pi x))\,dx = \left[\frac{x}{2} + \frac{\sin((2n+1)\pi x)}{2(2n+1)\pi}\right]_0^1 \overset{(1)}{=} \frac{1}{2}$. Integrating by parts twice we find that

$$\langle f,\varphi_n\rangle = \int_0^1 (1-x^2)\cos((n+\tfrac{1}{2})\pi x)\,dx = \cancel{\frac{(1-x^2)\sin((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})\pi}\Bigg|_{x=0}^{1}}^{\,0} - \int_0^1 (-2x)\frac{\sin((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})\pi}\,dx$$

$$= \frac{2}{(n+\frac{1}{2})\pi}\int_0^1 x\sin((n+\tfrac{1}{2})\pi x) = \frac{2}{(n+\frac{1}{2})\pi}\left[\cancel{\frac{-x\cos((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})\pi}\Bigg|_{x=0}^{1}}^{\,0} - \int_0^1 \frac{-\cos((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})\pi}\,dx\right]$$

$$= \frac{2}{(n+\frac{1}{2})^2\pi^2}\left(\frac{\sin((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})\pi}\Bigg|_{x=0}^{1}\right) = \frac{2\sin((n+\frac{1}{2})\pi)}{(n+\frac{1}{2})^3\pi^3} = \frac{2(-1)^n}{(n+\frac{1}{2})^3\pi^3}\cdot\frac{2^3}{2^3} = \frac{16(-1)^n}{(2n+1)^3\pi^3}. \quad (2)$$

Therefore $c_n = \frac{16(-1)^n/(2n+1)^3\pi^3}{1/2} \overset{(1)}{=} \frac{32(-1)^n}{(2n+1)^3\pi^3}$ if $n = 0,1,2,\dots$ so

$$f(x) \overset{(1)}{\sim} \sum_{n=0}^{\infty} \frac{32(-1)^n\cos((2n+1)\pi x/2)}{(2n+1)^3\pi^3}.$$

(b) Note that if $\varphi_n(x) = \cos\left(\frac{(2n+1)\pi x}{2}\right)$ then $-\frac{d^2}{dx^2}\varphi_n(x) = \frac{(2n+1)^2\pi^2}{2^2}\varphi_n(x)$ and $\varphi_n(1) = 0$ and $\varphi_n'(0) = 0$, so $\Phi = \{\varphi_n : n = 0, 1, 2, \ldots\}$ is an orthogonal set of eigenfunctions for the operator $T = -\frac{d^2}{dx^2}$ on $V = \{\varphi \in C^2[0,1] : \varphi'(0) = 0 = \varphi(1)\}$

(In problem 3 we will see that Φ a complete orthogonal set of eigenfunctions for T; we will assume this for now.) Observe that $f(x) = 1 - x^2$, $f'(x) = -2x$,

(2) and $f''(x) = -2$ and these functions are continuous on $[0,1]$. Furthermore

(2) $f(1) = 0$ and $f'(0) = 0$ so f satisfies the boundary conditions in V that

(2) generated the set of eigenfunctions Φ. By the Uniform Convergence Theorem 2 of Sec. 5.4, it follows that the Fourier series of f with respect to Φ converges uniformly to f on $[0,1]$.

(4) (c) Because uniform convergence implies pointwise convergence, it follows from part (b) that the Fourier series of f with respect to Φ converges pointwise

(2) to f on $[0,1]$.

(3) (d) Because uniform convergence implies L^2-convergence, it follows from part (b) that the Fourier series of f with respect to Φ converges in the L^2-sense

(2) to f on $[0,1]$.

(e) By part (c), $f(x) = \sum_{n=0}^{\infty} \frac{32(-1)^n \cos((2n+1)\pi x/2)}{(2n+1)^3\pi^3}$ for all $0 \le x \le 1$. Taking

(2) $x = 0$ in this identity gives $1 = f(0) = \sum_{n=0}^{\infty} \frac{32(-1)^n}{(2n+1)^3\pi^3}$ so $\boxed{\frac{\pi^3}{32} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}}$

(f) By Parseval's identity, $\sum_{n=0}^{\infty} \left|\frac{32(-1)^n}{(2n+1)^3\pi^3}\right|^2 \int_0^1 \cos^2\left(\frac{(2n+1)\pi x}{2}\right)dx = \int_0^1 (1-x^2)^2\,dx$

and routine calculations then yield

$$\frac{(32)^2}{\pi^6}\cdot\frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6} = \frac{8}{15} \quad \text{so} \quad \boxed{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6} = \frac{\pi^6}{960}}.$$

(1)

3.(35 pts.) Solve $u_{tt}-u_{xx}\overset{(1)}{=}0$ on $0<x<1$, $0<t<\infty$, subject to $u_x(0,t)\overset{(2)}{=}0\overset{(3)}{=}u(1,t)$ if $t\ge 0$ and $u(x,0)\underset{(4)}{=}1-x^2$, $u_t(x,0)\underset{(5)}{=}0$ if $0\le x\le 1$.

We use the separation of variables method. We seek nontrivial solutions to the homogeneous portion of the problem, (1)-(2)-(3)-(5), of the form $u(x,t)=X(x)T(t)$. Substituting in (1) gives $X(x)T''(t)-X''(x)T(t)=0$ so $\dfrac{-X''(x)}{X(x)}=\dfrac{-T''(t)}{T(t)}=\text{constant}=\lambda$. Substituting in (2) yields $X'(0)T(t)=0$ if $t\ge 0$, and nontriviality of $u(x,t)=X(x)T(t)$ implies $X'(0)=0$. Arguing similarly with (3) and (5) leads to $X(1)=0$ and $T'(0)=0$. Thus we have the system of coupled ODEs and boundary conditions:

(8) $\quad\begin{cases} X''(x)+\lambda X(x)\overset{(6)}{=}0, \quad X'(0)\overset{(7)}{=}0\overset{(8)}{=}X(1), \\ T''(t)+\lambda T(t)\overset{(9)}{=}0, \quad T'(0)\overset{(10)}{=}0. \end{cases}$ (4)

Note that (6)-(7)-(8) is the eigenvalue problem for the symmetric operator $T=-\dfrac{d^2}{dx^2}$ on $V=\{\varphi\in C^2[0,1]:\ \varphi'(0)=0=\varphi(1)\}$. According to problem 1, the eigenvalues are

(2) real and nonnegative.

Case $\lambda=0$: The general solution of (6) in this case, $X''(x)=0$, is $X(x)=c_1x+c_2$. Then $0=X'(0)=c_1$ and $0=X(1)=c_1+c_2$ so $c_1=0=c_2$. That is, all solutions

(2) to (6)-(7)-(8) in this case are trivial so 0 is not an eigenvalue.

Case $\lambda>0$, say $\lambda=\alpha^2$ where $\alpha>0$. The general solution of (6) in this case, $X''(x)+\alpha^2X(x)=0$, is $X(x)=c_1\cos(\alpha x)+c_2\sin(\alpha x)$. Then $X'(x)=-\alpha c_1\sin(\alpha x)+\alpha c_2\cos(\alpha x)$ so $0=X'(0)=\alpha c_2$ implies $c_2=0$. Also $0=X(1)=c_1\cos(\alpha)+c_2\sin(\alpha)$

(2) $=c_1\cos(\alpha)$ so the condition for a nontrivial solution is $\cos(\alpha)=0$. Hence $\alpha=\alpha_n=\dfrac{(2n+1)\pi}{2}$ $(n=0,1,2,\ldots)$. Consequently the eigenvalues and eigenfunctions of (6)-(7)-(8) are, respectively,

$$\lambda_n=\alpha_n^2=\frac{(2n+1)^2\pi^2}{4} \quad\text{and}\quad X_n(x)=\cos(\alpha_n x)=\cos\left(\frac{(2n+1)\pi x}{2}\right)\ (n=0,1,2,\ldots$$

(2[illegible] points on this page.) (2) (2)

Substituting $\lambda = \lambda_n = \frac{(2n+1)^2\pi^2}{4}$ in (9) gives $T_n''(t) + \frac{(2n+1)^2\pi^2}{4} T_n(t) = 0$, which has general solution $T_n(t) = c_1\cos\left((2n+1)\frac{\pi t}{2}\right) + c_2\sin\left((2n+1)\frac{\pi t}{2}\right)$. Then (10) gives $0 = T_n'(0) = (2n+1)\frac{\pi}{2} c_2$ so $c_2 = 0$. That is, $T_n(t) = \cos\left(\frac{(2n+1)\pi t}{2}\right)$

(2) up to a constant factor.

(2) Observe that $u_n(x,t) = X_n(x)T_n(t) = \cos\left((2n+1)\frac{\pi x}{2}\right)\cos\left((2n+1)\frac{\pi t}{2}\right)$ solves (1)-(2)-(3)-(5) and consequently a formal solution to (1)-(2)-(3)-(5) is

(4) (11) $$u(x,t) = \sum_{n=0}^{\infty} c_n \cos\left((2n+1)\frac{\pi x}{2}\right)\cos\left((2n+1)\frac{\pi t}{2}\right)$$

where $c_0, c_1, c_2, \ldots$ are arbitrary constants. To solve (4) as well, we need

(1) (12) $$1 - x^2 = u(x,0) = \sum_{n=0}^{\infty} c_n \cos\left((2n+1)\frac{\pi x}{2}\right) \quad \text{if } 0 \le x \le 1.$$

If we choose the c_n's to be the Fourier coefficients of $f(x) = 1 - x^2$ with

(2) respect to $\Phi = \left\{\cos\left((2n+1)\frac{\pi x}{2}\right)\right\}_{n=0}^{\infty}$ on $[0,1]$, then problem 2 shows that (12) holds. Thus

(2) $$\boxed{u(x,t) = \sum_{n=0}^{\infty} \frac{32(-1)^n \cos((2n+1)\pi x/2)\cos((2n+1)\pi t/2)}{(2n+1)^3\pi^3}}$$

solves (1)-(2)-(3)-(4)-(5).

Note: Energy methods can be used to show that this solution is unique.

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Convergence Theorems

Consider the eigenvalue problem

(1) $X''(x)+\lambda X(x)=0$ in $a<x<b$ with any symmetric boundary conditions

and let $\Phi=\{X_1,X_2,X_3,\ldots\}$ be a complete orthogonal set of eigenfunctions for (1). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$\sum_{n=1}^{\infty} A_n X_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle} \quad (n=1,2,3,\ldots).$$

Theorem 2. (Uniform Convergence) If
(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and
(ii) f satisfies the given symmetric boundary conditions,
then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 – Convergence) If

$$\int_a^b |f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)
(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.
(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty} A_n X_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4 ∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x.

Math 325

Exam III

Fall 2012

$n = 32$

median = 65.5

mean = 67.0

standard deviation = 22.1

Distribution of Scores:

Range	Graduate Grade	Undergraduate Grade	Frequency
87 - 100	A	A	7
73 - 86	B	B	6
60 - 72	C	B	9
50 - 59	C	C	3
0 - 49	F	D	7