Mathematics 325 | Exam III Summer 2009 | Name: Dr. Grow (1 pt.)

1.(33 pts.) (a) Show that the Fourier sine series of $f(x)=x(\pi-x)$ on $[0,\pi]$ is $\displaystyle\sum_{k=0}^{\infty}\frac{8\sin((2k+1)x)}{\pi(2k+1)^3}$.

(b) On the same coordinate axes, sketch the graph of f and the sum of the first three nonzero terms of the Fourier sine series of f on $[0,\pi]$.

(c) Based on the graphs in part (b), does it appear that the Fourier sine series of f converges uniformly to f on $[0,\pi]$?

(d) Assuming that the Fourier sine series of f converges pointwise to f on $[0,\pi]$, use the results above to find the sum of the infinite series $1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots$.

16 (a) $$b_n=\frac{\langle f,\sin(n\cdot)\rangle}{\langle\sin(n\cdot),\sin(n\cdot)\rangle}=\frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx=\frac{2}{\pi}\int_0^\pi \overbrace{x(\pi-x)}^{U}\overbrace{\sin(nx)dx}^{dV}=$$

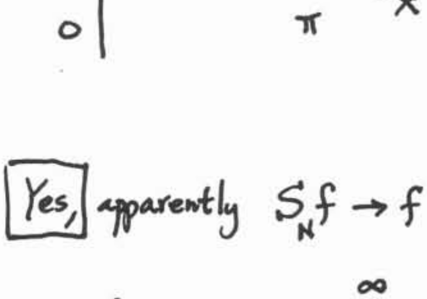
$$\cancel{-\frac{2}{\pi}x(\pi-x)\frac{\cos(nx)}{n}\Big|_0^\pi}^{\,0}+\frac{2}{n\pi}\int_0^\pi\overbrace{(\pi-2x)}^{U}\overbrace{\cos(nx)\,dx}^{dV}=\cancel{\frac{2}{n\pi}(\pi-2x)\frac{\sin(nx)}{n}\Big|_0^\pi}^{\,0}+\frac{2}{n^2\pi}\int_0^\pi\sin(nx)\,2\,dx$$

$$=\frac{-4}{n^2\pi}\cdot\frac{\cos(nx)}{n}\Big|_0^\pi=\frac{(-4)((-1)^n-1)}{\pi n^3}=\begin{cases}0 & \text{if } n=2k \text{ is even,}\\ \dfrac{8}{\pi(2k+1)^3} & \text{if } n=2k+1 \text{ is odd.}\end{cases}$$

Therefore the Fourier sine series of f is

$$\sum_{n=1}^{\infty}b_n\sin(nx)=\sum_{\substack{n=1\\(n\text{ odd})}}^{\infty}b_n\sin(nx)=\boxed{\sum_{k=0}^{\infty}\frac{8\sin((2k+1)x)}{\pi(2k+1)^3}}.$$

6 (b)



$y=f(x)=x(\pi-x)$

$y=S_5f(x)=\frac{8}{\pi}\left(\sin(x)+\frac{\sin(3x)}{27}+\frac{\sin(5x)}{125}\right)$

(To the resolving power of an HP-49G, the graphs of f and S_5f are indistinguishable on $[0,\pi]$.)

3 (c) $\boxed{\text{Yes,}}$ apparently $S_Nf\to f$ uniformly on $[0,\pi]$ as $N\to\infty$.

8 (d) $$\frac{\pi^2}{4}=f\left(\frac{\pi}{2}\right)=\sum_{k=0}^{\infty}\frac{8\overbrace{\sin((2k+1)\pi/2)}^{(-1)^k}}{\pi(2k+1)^3}\quad\text{so}\quad\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}=\frac{\pi^2}{4}\cdot\frac{\pi}{8}=\boxed{\frac{\pi^3}{32}}$$

2.(33 pts.) (a) Find a solution to $u_{tt} - u_{xx} \overset{①}{=} 0$ in the strip $0 < x < \pi$ and $0 < t < \infty$ satisfying the boundary conditions $u(0,t) \overset{②}{=} 0$ and $u(\pi,t) \overset{③}{=} 0$ for $t \geq 0$ and the initial conditions $u(x,0) \overset{④}{=} x(\pi - x)$ and $u_t(x,0) \overset{⑤}{=} 0$ for $0 \leq x \leq \pi$. (Hint: You may find the results of problem 1 useful.)

(b) Is the solution to the problem in part (a) unique? Justify your answer.

20 pts.

(a) We use separation of variables. We seek nontrivial solutions to the homogeneous part

2 of the problem, ①-②-③-⑤, of the form $u(x,t) = X(x)T(t)$. Substituting this functional form in ① and rearranging yields $-\frac{T''(t)}{T(t)} = -\frac{X''(x)}{X(x)} = \text{constant} = \lambda$. Also, substituting in ②, ③, and ⑤ and using the fact that u is not identically zero gives $X(0) = 0 = X(\pi) = T'(0)$

Therefore we are lead to the coupled system of B.V.P.'s:

Eigenvalue Problem

5 $$\left\{ \begin{array}{l} \boxed{X''(x) + \lambda X(x) = 0, \quad X(0) = 0 = X(\pi)} \\ T''(t) + \lambda T(t) = 0, \quad T'(0) = 0 \end{array} \right.$$

7

From work done in class and on homework, the Dirichlet B.C.'s in the eigenvalue problem lead to:

9 $$\left. \begin{array}{l} \text{Eigenvalues: } \lambda_n = n^2 \\ \text{Eigenfunctions: } X_n(x) = \sin(nx) \end{array} \right\} n = 1, 2, 3, \ldots$$

1'

The t-equations then become $T_n''(t) + n^2 T(t) = 0$, $T_n'(0) = 0$ with solution

13 $T_n(t) = \cos(nt)$ (up to a constant factor). Thus $u_n(x,t) = X_n(x)T_n(t) = \sin(nx)\cos(nt)$ solves ①-②-③-⑤ for $n = 1, 2, 3, \ldots$ By the superposition principle,

15 $$(*) \quad u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\cos(nt)$$

solves ①-②-③-⑤ formally for arbitrary constants $b_1, b_2, b_3, \ldots$ We need to choose the constants so $(*)$ satisfies ④; i.e.

17 $$x(\pi - x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{for all } 0 \leq x \leq \pi.$$

By problem 1, we should choose $b_n = \begin{cases} 0 & \text{if } n = 2k \text{ is even,} \\ \frac{8}{\pi(2k+1)^3} & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$

Therefore

20 $$\boxed{u(x,t) = \sum_{k=0}^{\infty} \frac{8}{\pi(2k+1)^3} \sin((2k+1)x)\cos((2k+1)t)}$$ solves ①-②-③-④-⑤.

(OVER)

13 pts.

(b) Suppose there were another solution $u = v(x,t)$ to ①-②-③-④-⑤.

2 Therefore $w(x,t) = u(x,t) - v(x,t)$ would solve

$$
\begin{array}{lll}
(1') & w_{tt} - w_{xx} = 0 & \text{if } 0 < x < \pi \text{ and } 0 < t < \infty, \\
(2')\text{-}(3') & w(0,t) = 0 = w(\pi,t) & \text{if } t \geq 0, \\
(4')\text{-}(5') & w(x,0) = 0 = w_t(x,0) & \text{if } 0 \leq x \leq \pi .
\end{array}
$$

4

6 Let $E(t) = \int_0^\pi [w_t^2(x,t) + w_x^2(x,t)]dx$ be the energy of the solution to (1')-(2')-(3')-(4')-(5')

Then $\frac{dE}{dt} = \frac{d}{dt}\int_0^\pi (w_t^2 + w_x^2)dx = \int_0^\pi \frac{\partial}{\partial t}(w_t^2 + w_x^2)dx = 2\int_0^\pi (w_t w_{tt} + w_x w_{xt})dx$

$$\overset{(1')}{=} 2\int_0^\pi \overbrace{w_t}^{U} \overbrace{w_{xx}dx}^{dV} + 2\int_0^\pi w_x w_{xt}\, dx = 2w_t(x,t)w_x(x,t)\Big|_{x=0}^{\pi} - \cancel{2\int_0^\pi w_x w_{tx}dx} + \cancel{2\int_0^\pi w_x w_{xt}dx}$$

By (2')-(3'), $w_t(\pi,t) = \lim_{k\to 0} \frac{w(\pi,t+k) - w(\pi,t)}{k} = 0$ and $w_t(0,t) = \lim_{k\to 0} \frac{w(0,t+k) - w(0,t)}{k}$ $= 0$. Therefore

8
$$\frac{dE}{dt} = 2w_t(\pi,t)w_x(\pi,t) - 2w_t(0,t)w_x(0,t) = 0$$

Consequently $E(t) = E(0)$ for all $t \geq 0$. But $E(0) = \int_0^\pi (w_t^2(x,0) + w_x^2(x,0))dx$

10 $= 0$ by (4')-(5'). Therefore $E(t) = 0$ for all $t \geq 0$. By the vanishing theorem, for each $t \geq 0$, we have $w_t^2(x,t) + w_x^2(x,t) = 0$ for all $0 \leq x \leq \pi$. Consequently $w_t(x,t) = 0 = w_x(x,t)$ for all (x,t) in the strip $0 \leq x \leq \pi$, $0 \leq t < \infty$. It follows that $w(x,t) =$ constant in the strip. But (2'), (3'), or (4') implies $w(x,t) = 0$

13 in the strip. That is, $u(x,t) = v(x,t)$ in the strip so **the solution to ①-②-③-④-⑤ is unique**.

3.(33 pts.) Consider the operator $T = \dfrac{d^4}{dx^4}$ on the space

$$V = \left\{ f \in C^4[0,1] : f(0) = 0 = f(1) \text{ and } f''(0) = 0 = f''(1) \right\}.$$

(a) Is T symmetric on V equipped with the inner product $\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\,dx$? Justify your answer.

(b) Are all the eigenvalues of T real? Justify your answer.

(c) Are eigenfunctions of T, corresponding to distinct eigenvalues, orthogonal? Justify your answer.

(d) Are all the eigenvalues of T nonnegative? Justify your answer.

Bonus (10 pts.): Determine all the eigenvalues and eigenfunctions of T.

11 (a) Let f and g belong to V. Then $\langle Tf, g\rangle = \int_0^1 \underbrace{f^{(4)}(x)}_{dv} \overbrace{\overline{g(x)}}^{v}\,dx = \overline{g(x)} f^{(3)}(x)\Big|_0^1 - \int_0^1 \overline{g'(x)} f^{(3)}$

$= \left(\overline{g(x)} f'''(x) - \overline{g'(x)} f''(x)\right)\Big|_0^1 + \int_0^1 \underbrace{\overline{g''(x)}}_{v}\, \underbrace{f''(x)dx}_{dv} = \left(\overline{g(x)} f'''(x) - \overline{g'(x)} f''(x) + \overline{g''(x)} f'(x)\right)\Big|_0^1 - \int_0^1 \overline{g'''(x)}$

$= \left(\overline{g(x)} f'''(x) - \overline{g'(x)} f''(x) + \overline{g''(x)} f'(x) - \overline{g'''(x)} f(x)\right)\Big|_0^1 + \int_0^1 \overline{g^{(4)}(x)} f(x)\,dx.$

But $g(0) = 0 = g(1)$, $f''(0) = 0 = f''(1)$, $g''(0) = 0 = g''(1)$, and $f(0) = 0 = f(1)$ so

$$\langle Tf, g\rangle = \int_0^1 \overline{g^{(4)}(x)}\, f(x)\,dx = \langle f, Tg\rangle.$$

Therefore $\boxed{T \text{ is symmetric on } V.}$

5 (b) $\boxed{\text{Yes,}}$ by Theorem 2 in the lecture notes for Sec. 5.3, the eigenvalues of T are real.

5 (c) $\boxed{\text{Yes,}}$ by Theorem 1 in the lecture notes for Sec. 5.3, eigenfunctions of T corresponding to distinct eigenvalues are orthogonal on $[0,1]$.

12 (d) $\boxed{\text{Yes,}}$ all the eigenvalues of T are nonnegative. To see this we must adapt the proof of Theorem 3 in Sec. 5.3. (Theorem 3 itself does not apply directly because it is concerned with the operator $-\frac{d^2}{dx^2}$.) Let λ be an eigenvalue of $T = \frac{d^4}{dx^4}$ on V. Let f be an eigenfunction of T in V corresponding to λ. By taking the real and imaginary parts of f if necessary, we may assume that f is real-valued. Then

$\lambda\langle f, f\rangle = \langle \lambda f, f\rangle = \langle Tf, f\rangle = \int_0^1 f^{(4)}(x)\overline{f(x)}\,dx = \int_0^1 \underbrace{f(x)}_{v}\, \underbrace{f^{(4)}(x)dx}_{dv} =$

$f(x)f'''(x)\Big|_0^1 - \int_0^1 \underbrace{f'(x)}_{v}\, \underbrace{f'''(x)dx}_{dv} = \left(f(x)f'''(x) - f'(x)f''(x)\right)\Big|_0^1 + \int_0^1 (f''(x))^2\,dx.$

But $f \in V$ so $f(1) = 0 = f(0)$ and $f''(1) = 0 = f''(0)$ so

(OVER)

$$\lambda\langle f,f\rangle = \int_0^1 \left(f''(x)\right)^2 dx \geq 0 .$$

Since $\langle f,f\rangle > 0$ as well, it follows that $\lambda \geq 0$. (A little more thought shows that actually $\lambda > 0$.)

10 <u>Bonus</u>: Let λ be a (necessarily positive real) number and f a nonzero function in V such that $Tf = \lambda f$. Since $\lambda > 0$ we may write $\lambda = \beta^4$ where $\beta > 0$. Then

$$f^{(4)}(x) - \beta^4 f(x) \overset{1}{=} 0 \text{ if } 0<x<1 \text{ and } f(0) \overset{2}{=} 0 \overset{3}{=} f(1),\ f''(0) \overset{4}{=} 0 \overset{5}{=} f''(1).$$

Substituting $f(x) = e^{mx}$ in 1 yields $m^4 - \beta^4 = 0 \Rightarrow (m^2-\beta^2)(m^2+\beta^2) = 0 \Rightarrow m = \pm\beta, \pm i\beta$.

Therefore $f_1(x) = \cos(\beta x)$, $f_2(x) = \sin(\beta x)$, $f_3(x) = \cosh(\beta x)$, $f_4(x) = \sinh(\beta x)$ forms a fundamental set of solutions of 1 since their Wronskian is $W(f_1,f_2,f_3,f_4)(x) = 4\beta^6 \neq 0$ for all real x.

I.e. $f(x) = c_1\cos(\beta x) + c_2\sin(\beta x) + c_3\cosh(\beta x) + c_4\sinh(\beta x)$ is the general solution of 1.

Note that $f''(x) = -c_1\beta^2\cos(\beta x) - c_2\beta^2\sin(\beta x) + \beta^2 c_3\cosh(\beta x) + \beta^2 c_4\sinh(\beta x)$. Then

$$\left.\begin{aligned} 0 &\overset{2}{=} f(0) = c_1 + c_3 \\ 0 &\overset{4}{=} f''(0) = -c_1\beta^2 + c_3\beta^2 = (-c_1+c_3)\beta^2 \end{aligned}\right\} \Rightarrow c_1 = c_3 = 0 .$$

Therefore $f(x) = c_2\sin(\beta x) + c_4\sinh(\beta x)$ so

$$\left.\begin{aligned} 0 &\overset{3}{=} f(1) = c_2\sin(\beta) + c_4\sinh(\beta) \\ 0 &\overset{5}{=} f''(1) = -c_2\beta^2\sin(\beta) + c_4\beta^2\sinh(\beta) \end{aligned}\right\} \Rightarrow \begin{aligned} 0 &= 2c_2\beta^2\sin(\beta) \\ &\text{and} \\ 0 &= 2c_4\beta^2\sinh(\beta) . \end{aligned}$$

Since $\beta > 0$ and $\sinh(\beta) > 0$ it follows that $c_4 = 0$. In order for f to be nonzero we must therefore have $\sin(\beta) = 0$, so $\beta = \beta_n = n\pi$ $(n = 1,2,3,\ldots)$.

$$\boxed{\left.\begin{aligned} &\text{Eigenvalues of } T \text{ on } V: && \lambda_n = \beta_n^4 = n^4\pi^4 \\ &\text{Corresponding Eigenfunctions:} && f_n(x) = \sin(\beta_n x) = \sin(n\pi x) \end{aligned}\right\} n = 1,2,3,\ldots}$$

Math 325

Exam III

Summer 2009

number taking exam: 16

standard deviation: 19.5

mean: 77.5

Distribution of Scores:		frequency
87 – 100	A	5
73 – 86	B	6
60 – 72	C(graduate), B(undergraduate)	2
50 – 59	C	1
0 – 49	F(graduate), D(undergraduate)	2